- These slides/notes represent only part of the course, and were accompanied by face-to-face explanations on white-board and additional topics / learning materials.
- In preparation of these slides I have also benefited from various books and online material.
- Some of the slides contain animations which may not be visible in pdf version.
- Corrections, comments, feedback may be sent to https://www.linkedin.com/in/naveedrazzaqbutt/

ES 332
Signals and Systems

with

Dr. Naveed R. Butt

@

GIKI - FES

Introductions …

- Me
- You
- The Course

GIK Institute

BS in Engineering Sciences

[1998 - 2002]

Automation Engineer [2002 - 2004]

Riyadh Water Transportation System's SCADA upgrade project

KFUPM

MS Student/Staff [2004 - 2006]

- MS in Systems Engineering
- Thesis in nonlinear modelling & control
- Teaching (labs: DSP, Control)

LTH

[2006 - 2014]

- Positions: PhD Student/Staff, Postdoc, Research Associate
- PhD in Engineering (focus: Statistical Modelling & SP)
- Teaching + Research

Ericsson Research

[2014 - 2018]

- Senior Researcher
- Research + Patenting
- Next Generation WiFi & 5G

Jouf University [2018…]

Assistant Professor

College of Engineering

- Badminton, Bowling
- Weekend dinners
- Reading, Writing (poetry, short stories)

Roles

Teacher

Researcher

As a teacher

Full Courses

- Stochastic Processes
- Statistical DSP & Modelling
- Probabilistic Methods in Engineering
- Wave Propagation & Antennas
- Principles of Communications
- Digital Communications
- Satellite Communications
- Circuit Analysis II

Labs & Tutorials

- Time Series Analysis
- Signal Theory
- Advanced Control
- Modern Control Systems
- Digital Design

Supervision

Supervised and collaborated in various grad and postgrad theses.

As a researcher

Statistical Modelling of Signals & Systems

Spectroscopy
- Nuclear Quadrupole Resonance (NQR) signal detection
- Raman Signal Classification

Spectrum Estimation
- Missing Samples Cases
- Poly-spectra
- Coherence Spectra

Beamforming
- Radar & Sonar
- Pitch Estimation

Communications
- WiFi & 5G
- Antenna Arrays

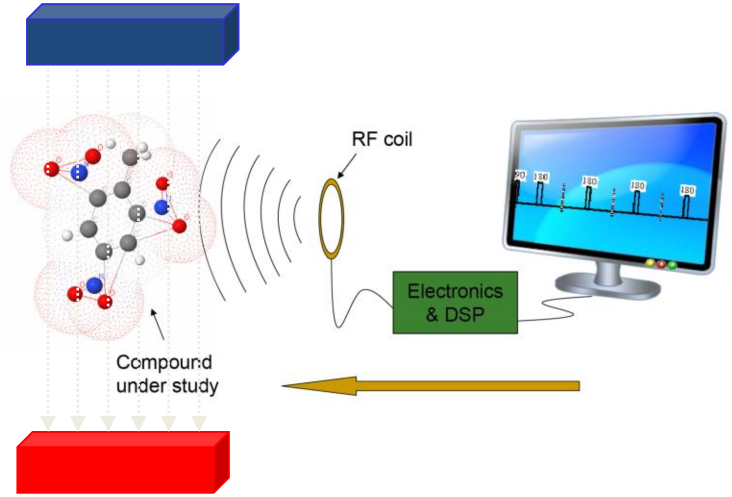
Control
- Nonlinear Plant Modelling & Control

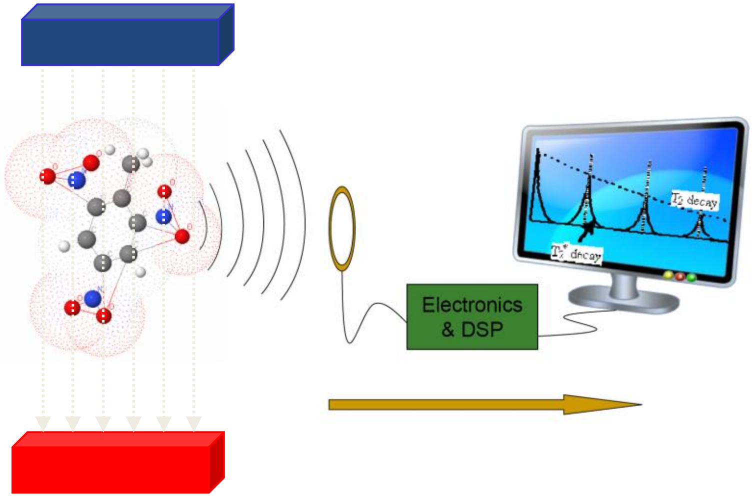
One of my research projects…

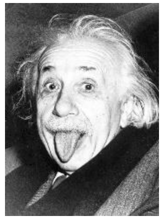
CONPHIRMER Project

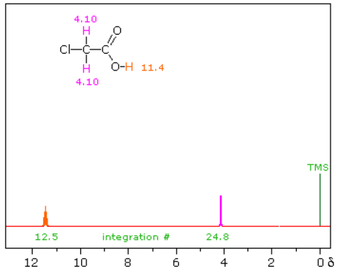
Question: how to quickly tell whether a medicine is fake?

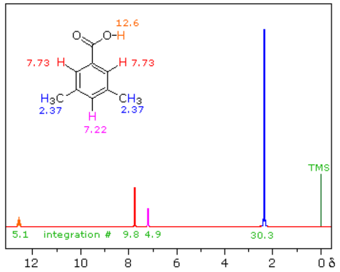
NMR vs. NQR


RF coil
Electronics & DSP
Compound under study

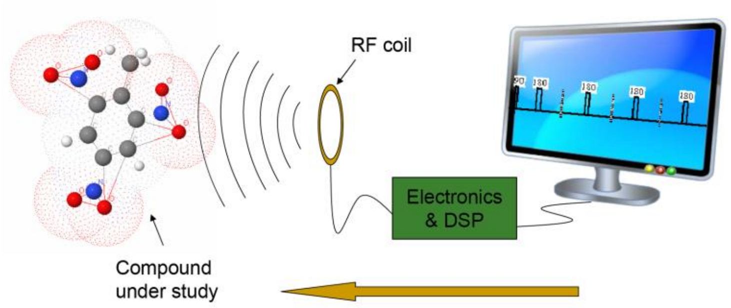

Electronics & DSP

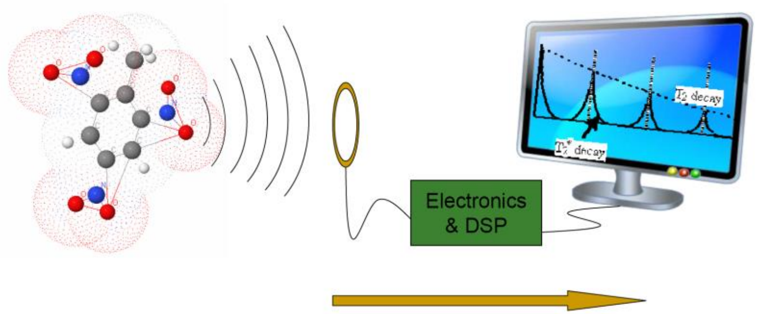


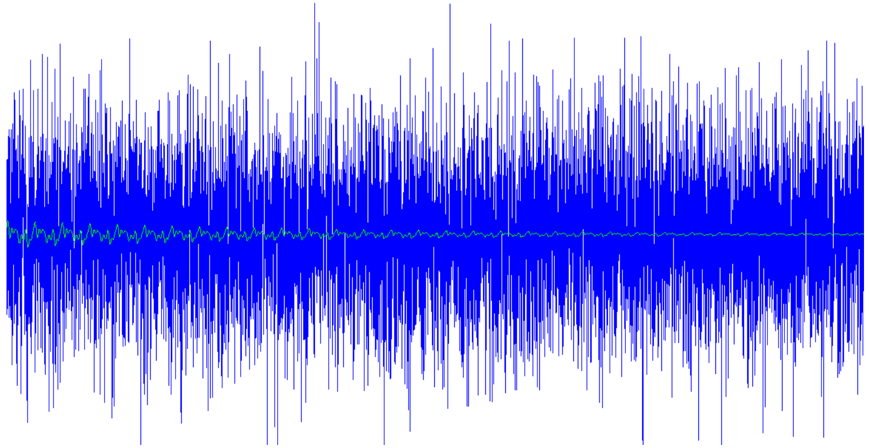

4.10
O-H 11.4
TMS
12.5
integration #
24.8
12 10 8 6 4 2 0 δ


12.6
7.73
7.73
2.37
2.37
7.22
TMS
5.1
integration #
9.8 4.9
30.3
12 10 8 6 4 2 0 δ

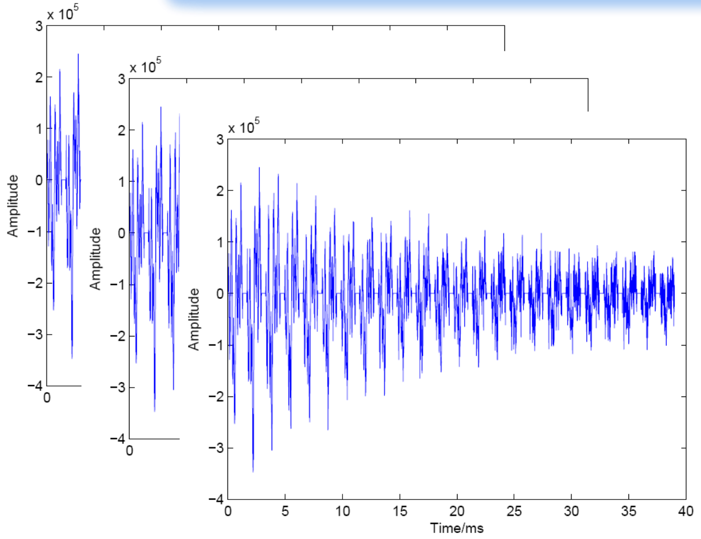
NQR Signal

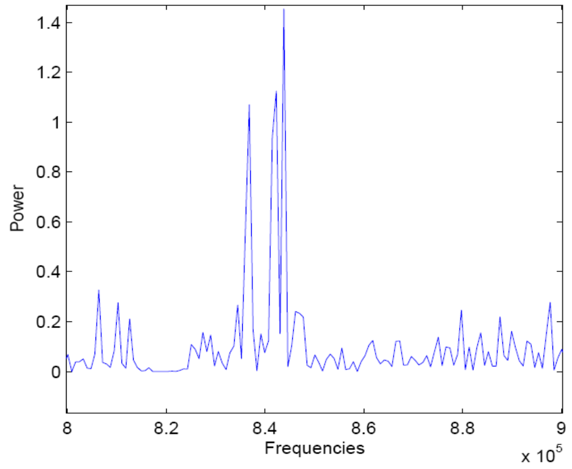

RF coil
Electronics
& DSP
Compound
under study


Electronics
& DSP



NQR Signal Modeling

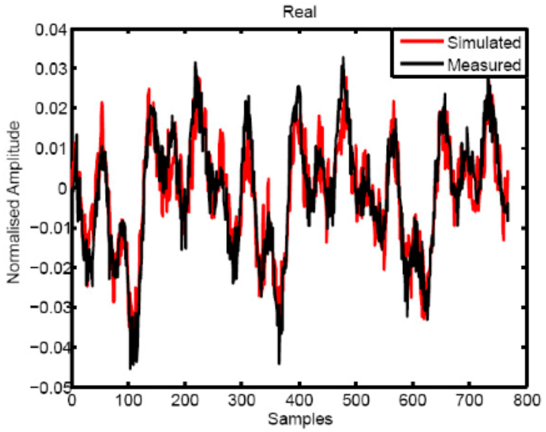




$$y_m^{(p)}(t) = \sum_{k=1}^{d^{(p)}} \alpha_k^{(p)} e^{-(t+m\mu)\eta_k^{(p)}} e^{-\beta_k^{(p)}|t-t_{sp}|+i\omega_k^{(p)}(T)t}$$

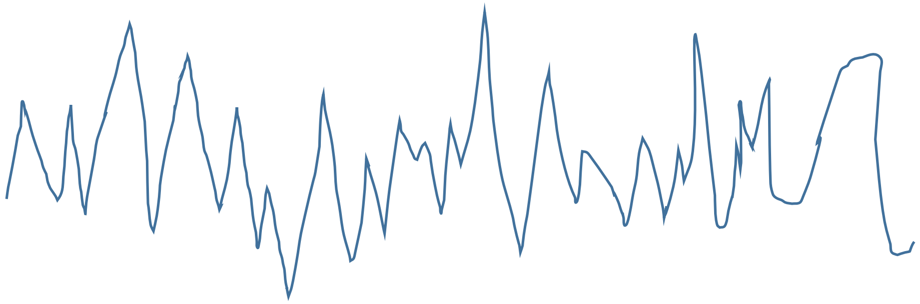
$$\omega_k^{(p)}(T) = a_k^{(p)} - b_k^{(p)} T$$

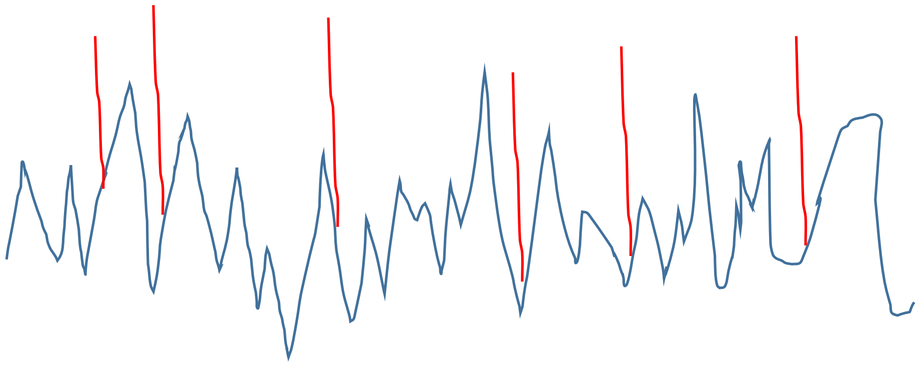
$$\alpha_k^{(p)} = \rho \kappa_k^{(p)} \qquad \left\| \boldsymbol{\kappa} - \boldsymbol{\kappa_a} \right\|_2^2 \leq \epsilon$$



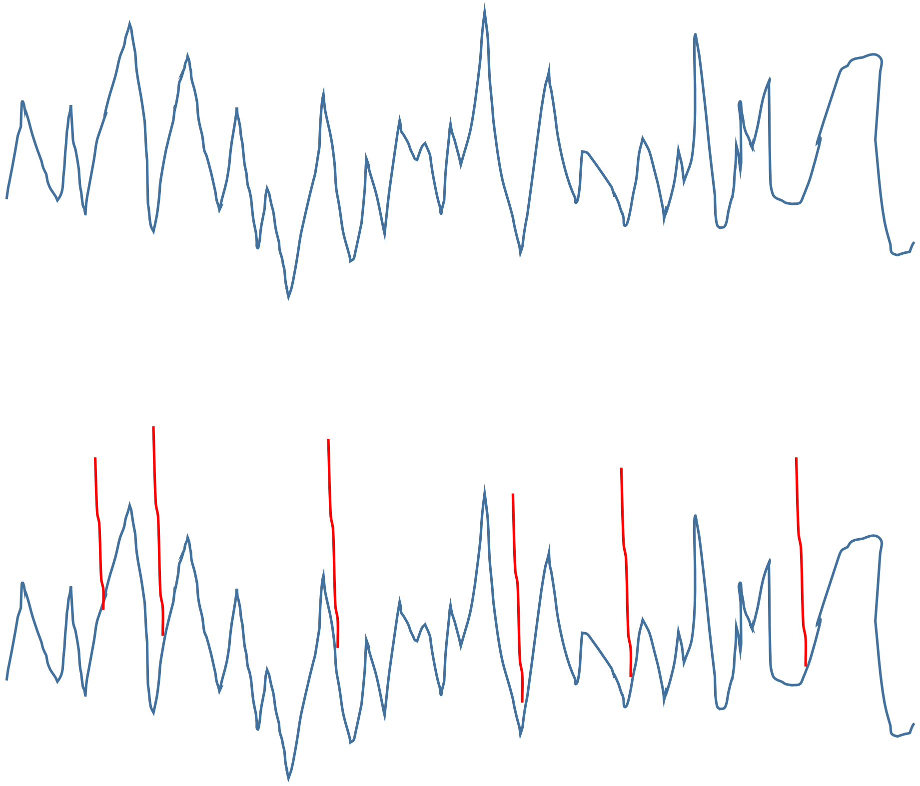
Introductions …

- Me
- You
- The Course

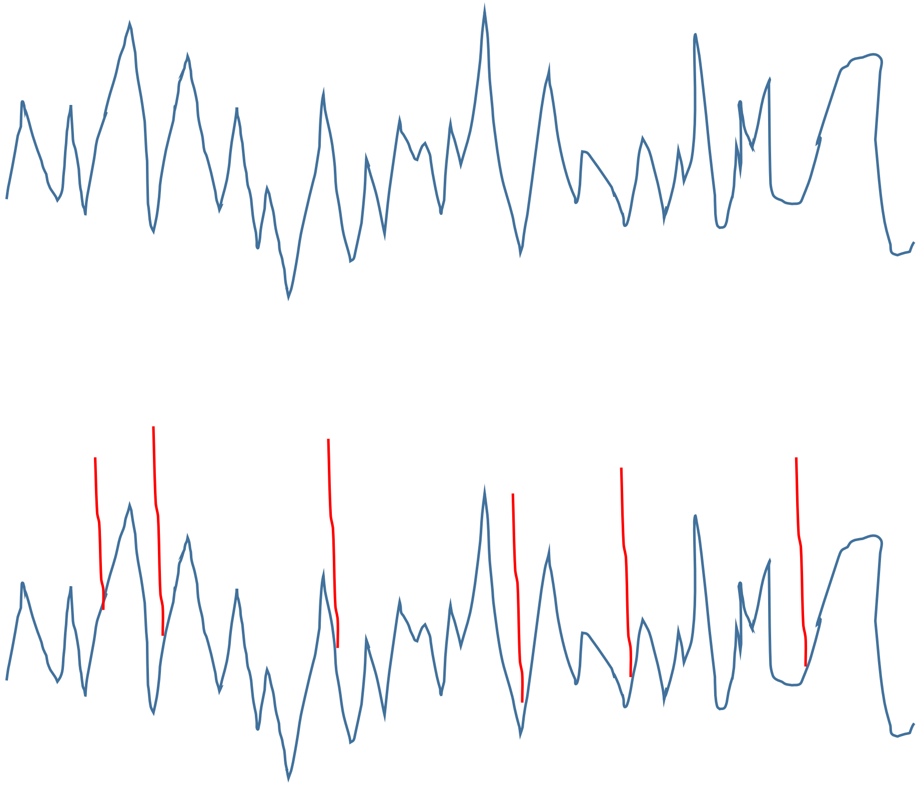


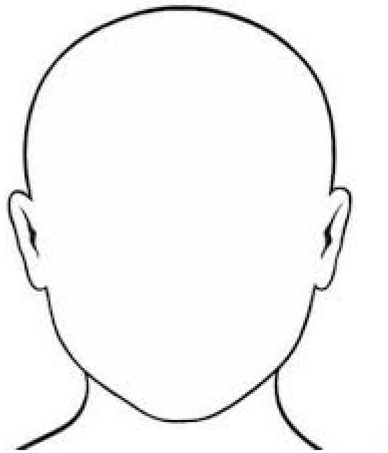


I hate this course



I hate this course

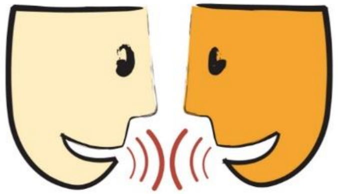


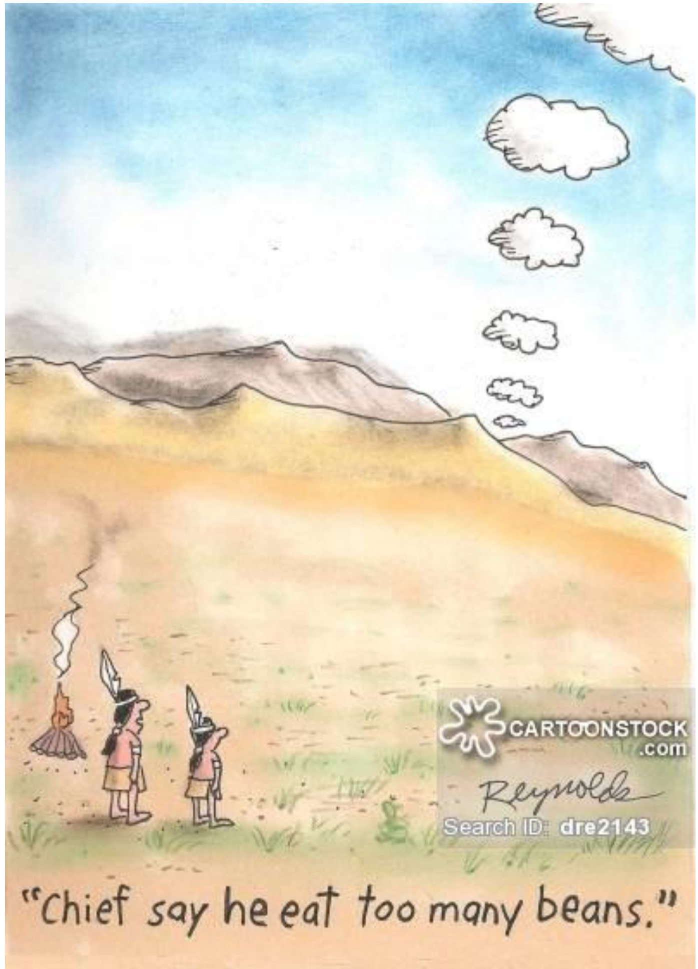


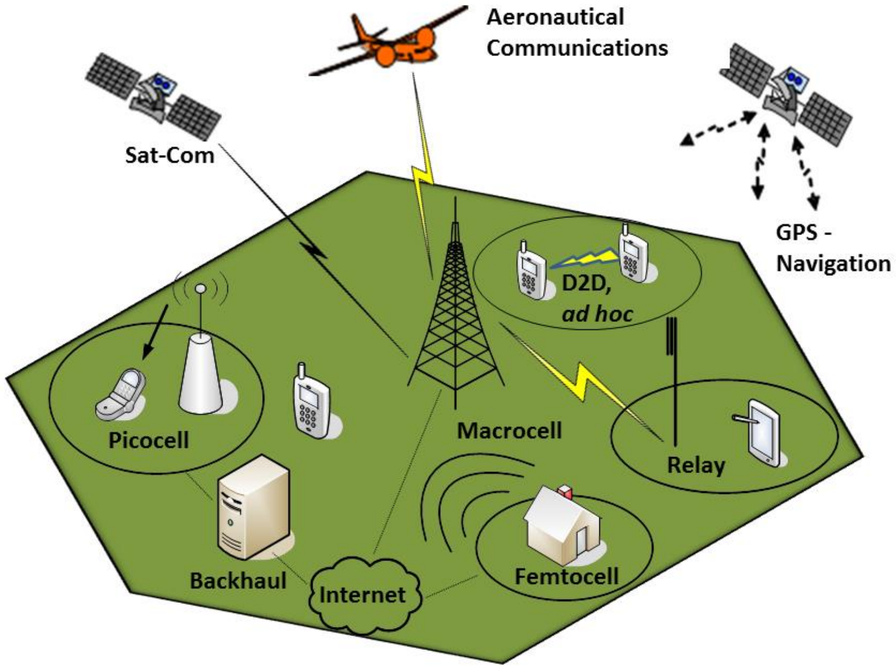
What is a signal and what is a system? (from engineering mathematics perspective)

Chapter 1

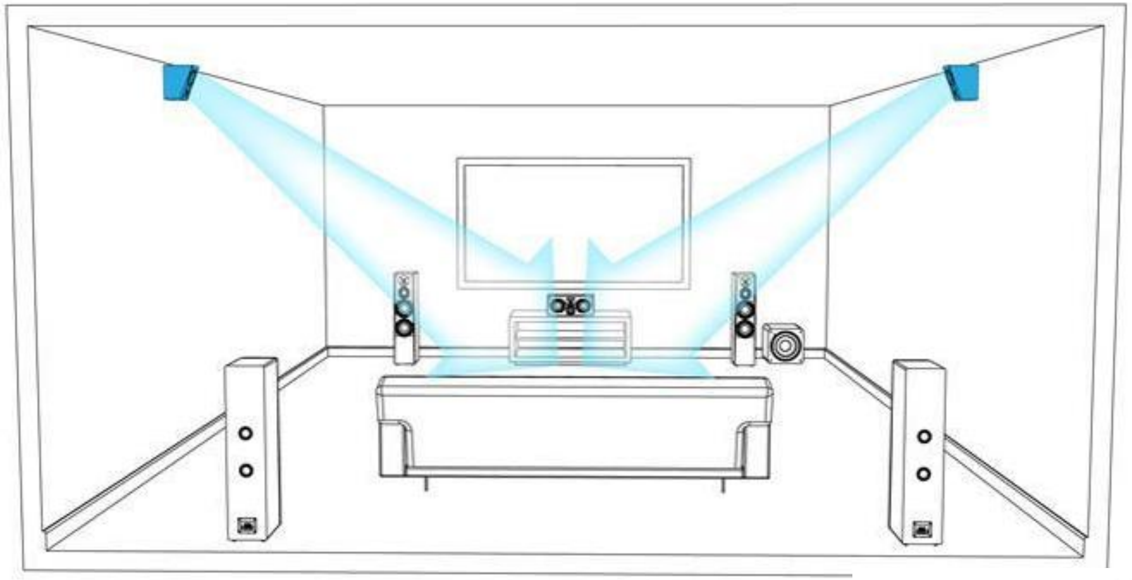
Signals

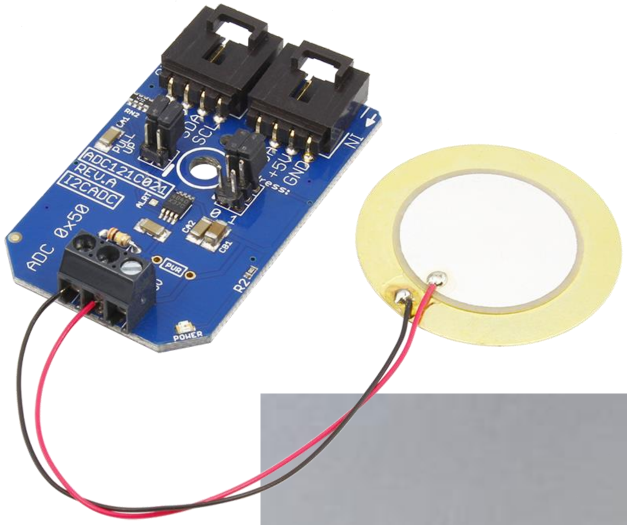


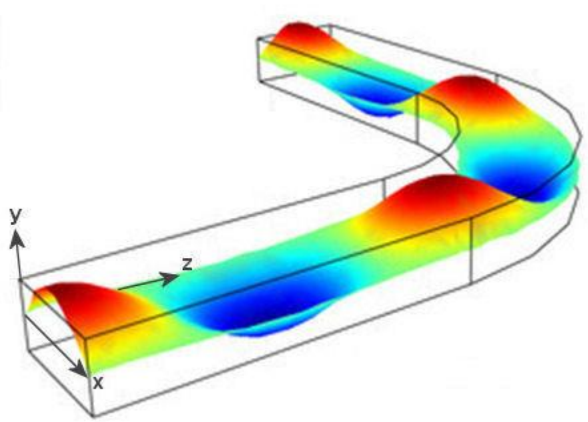

CARTOONSTOCK
.com
Search ID: dre2143
"Chief say he eat too many beans."


Aeronautical
Communications
Sat-Com
GPS -
Navigation
D2D,
ad hoc
Picocell
Macrocell
Relay
Backhaul
Internet
Femtocell

Systems

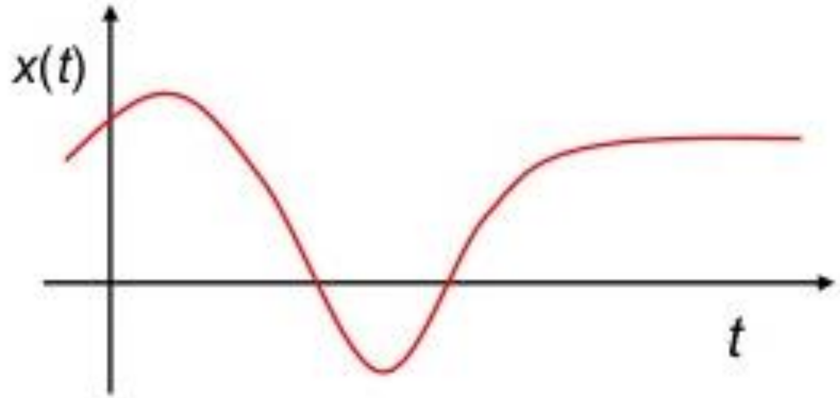


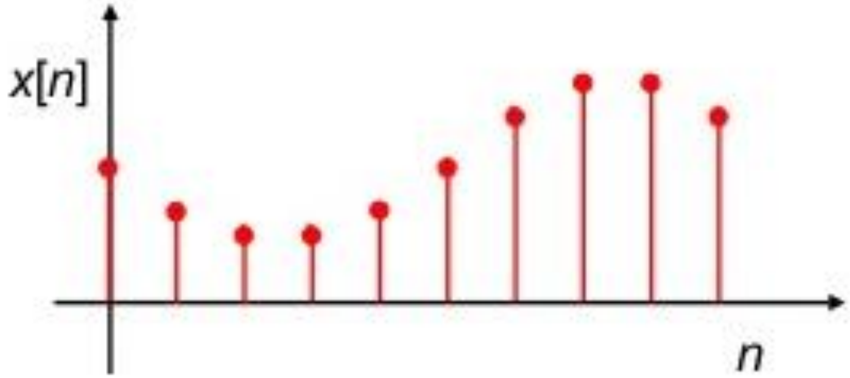




What are some of the common types of signals and systems?

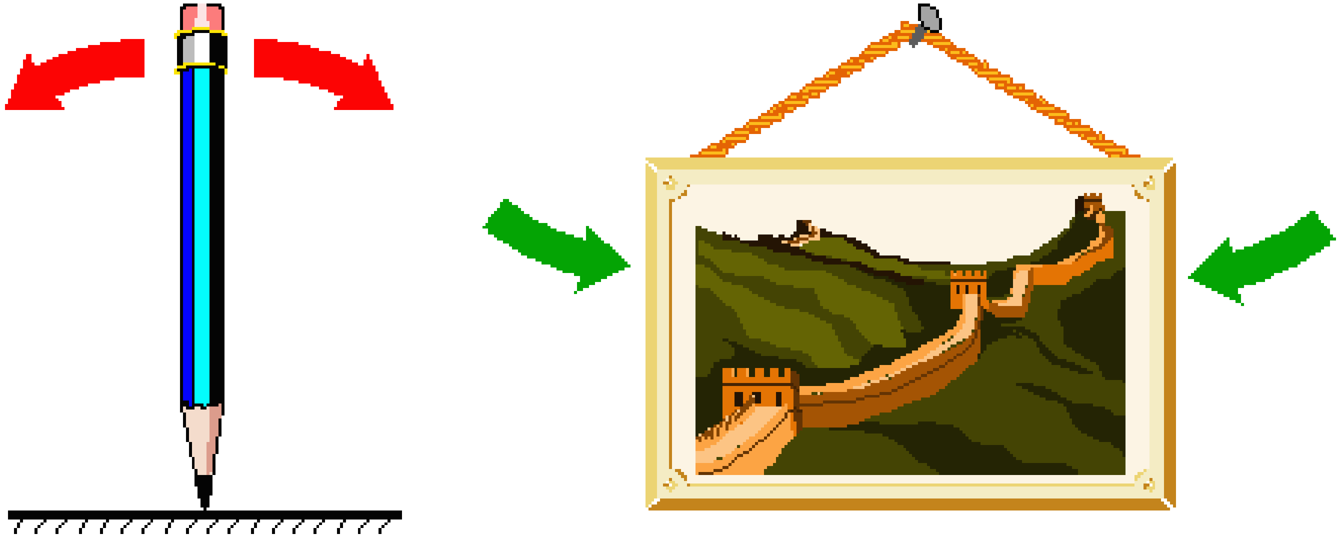
Chapter 1


x(t)
t


x[n]
n

What are some of the useful properties of signals and systems?

Chapter 1



Four major ways of modelling and analyzing signals and systems.

Time Domain	**Chapters 2, 3**
Laplace	**Chapter 4**
Z-Transform	**Chapter 5**
Fourier	**Chapters 6, 7, 9**

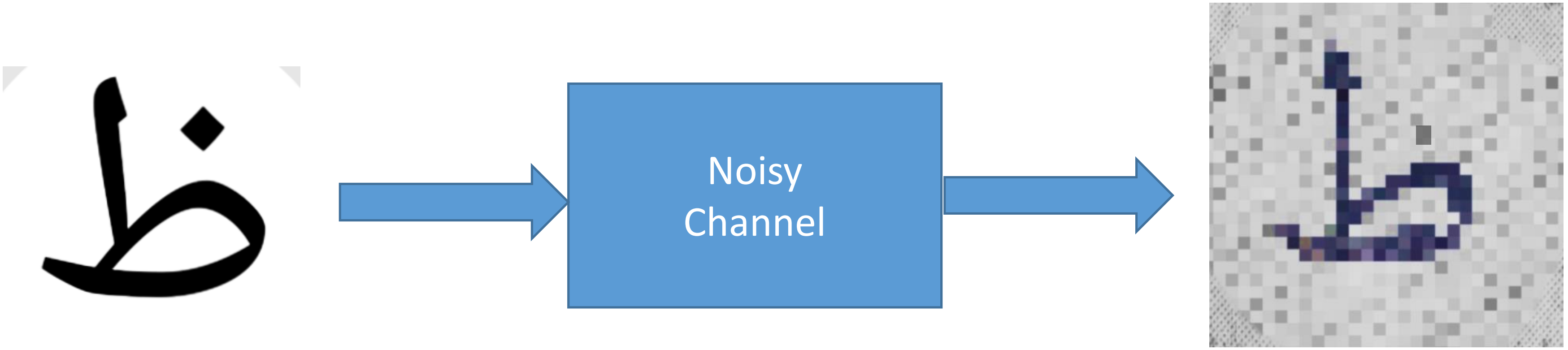
.9998
.9994
.9986
.9976
.9962
.9945
.9925
.9903
.9877
.9848
.9816
.9781
.9744
.9703
.9659
.9613
.9563
.9511
.9455
.9397
.9336
.9272
.9205
.9135
.9063
.8988
.8910

.6947
.6820
.6691
.6561
.6428
.6293
.6157
.6018
.5878
.5736
.5592
.5446
.5299
.5150
.5000
.4848
.4695
.4540
.4384
.4226
.4067

$$\cos\theta$$

How do signals and systems interact?

Chapters 2-7

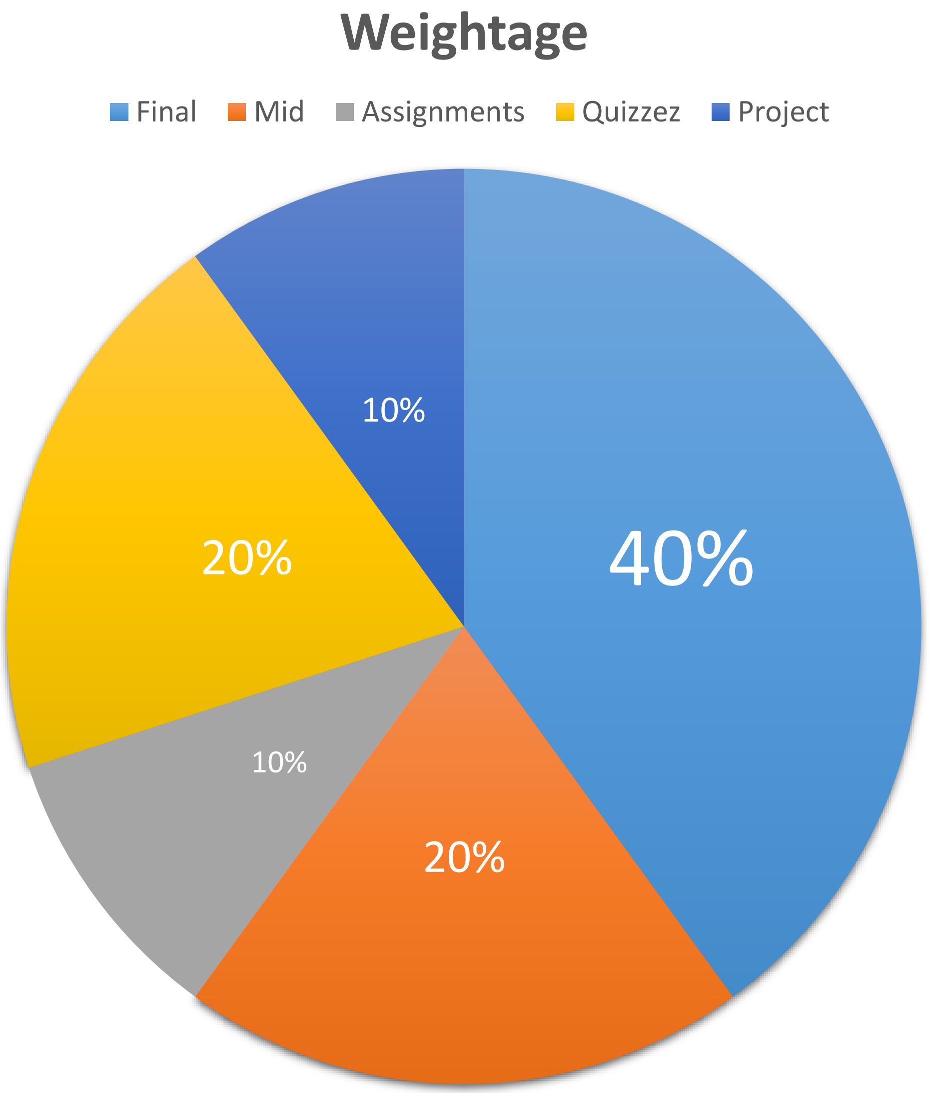

Noisy
Channel

Important Business!!

- 80% attendance is mandatory!
- Textbook
  - Lathi, B. P., and Green R. A., Linear Systems and Signals (3rd ed.), NY: Oxford University Press (2018)
- Contact
  - naveed.butt@giki.edu.pk
  - office: FES G-13

Learning Plan

- **Lectures**
  - Help discover and grasp new concepts
- **Quizzes & Assignments**
  - Help prepare/revise each week’s concepts
  - Keep you from lagging behind in course
- **Presentation**
  - Helps learn independent work & presentation
  - Prepares for final year project
- **Exams (Mid-1, Final)**
  - Help prepare entire course material



Course Learning Objectives (CLOs)

CLO #	Domain	Description	Assessment
CLO 1	Cognitive/Applying	**Apply** the basic knowledge of signals and systems to categorize and solve basic operations of signals and systems.	Quiz, Mid, Assign., Final
CLO 2	Cognitive/Applying	**Calculate** parameters related to continuous-time and discrete-time signals and systems in the time domain.	Quiz, Mid, Assign., Final
CLO 3	Cognitive/Analyzing	**Analyze** continuous-time and discrete-time signals and systems in the transform domains including Laplace, Fourier, and Z transforms.	Quiz, Mid, Assign., Final
CLO 4	Communication	**Demonstrate** the ability to review/implement material related to signals and systems and formally present the results.	Presentation

Questions?? Thoughts??

ES 332
Signals and Systems

with

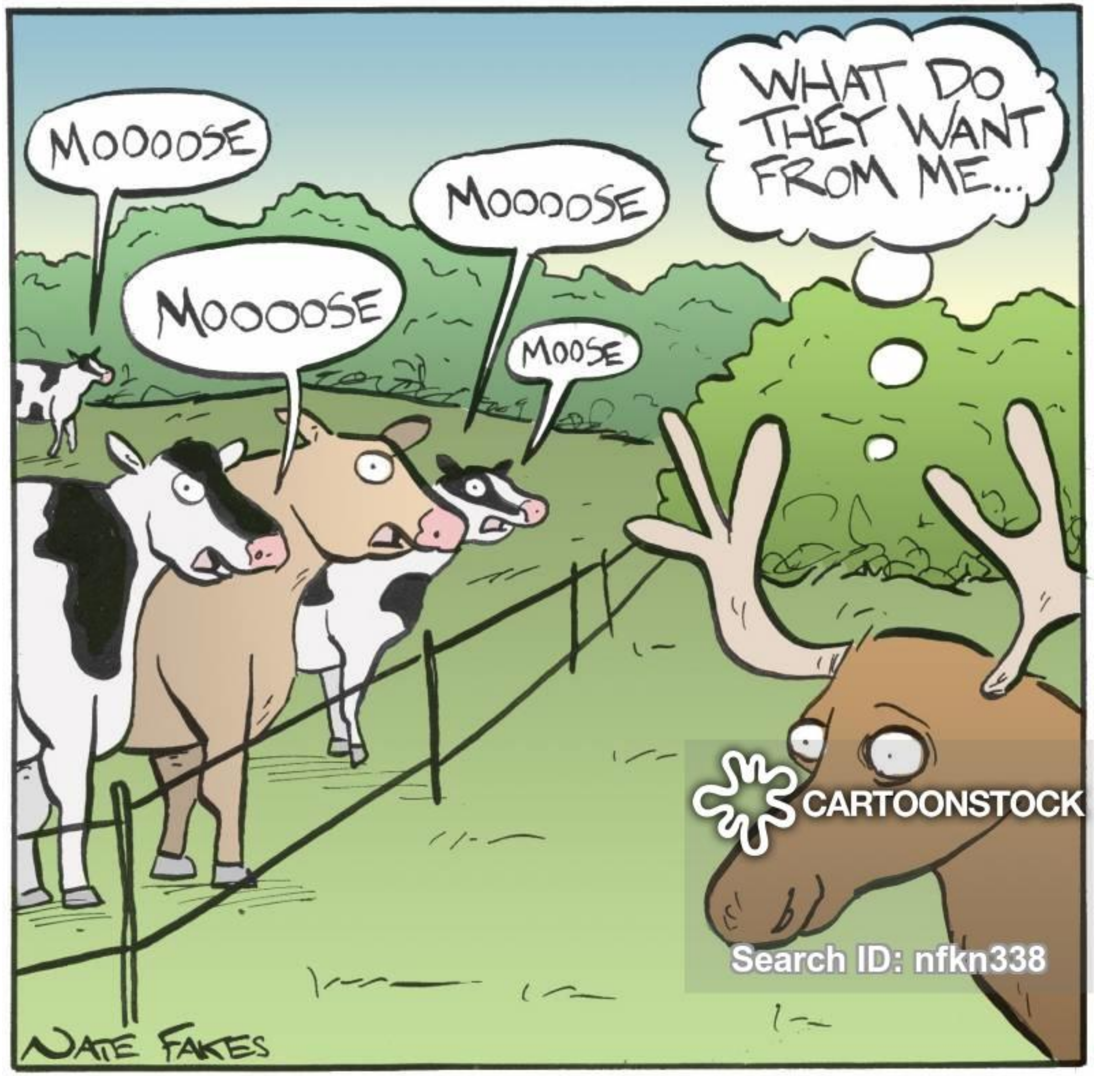
Dr. Naveed R. Butt

@

GIKI - FES

Vocabulary

- زبانِ یارِ مَن تُرکی

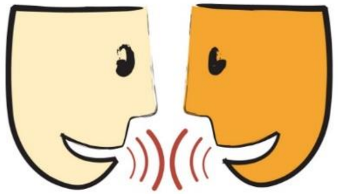


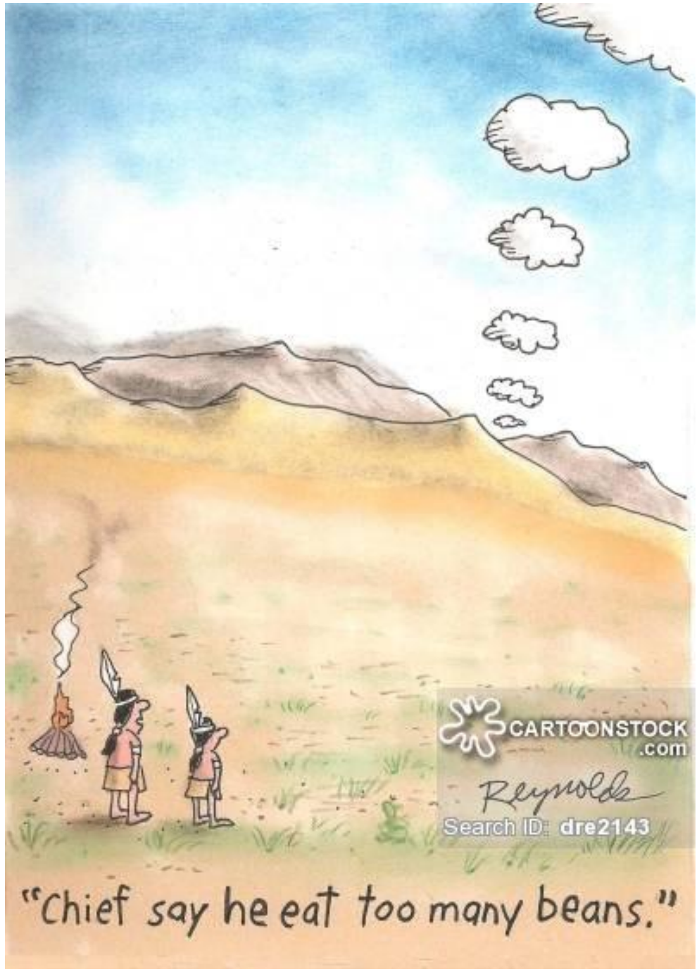
What a moose hears

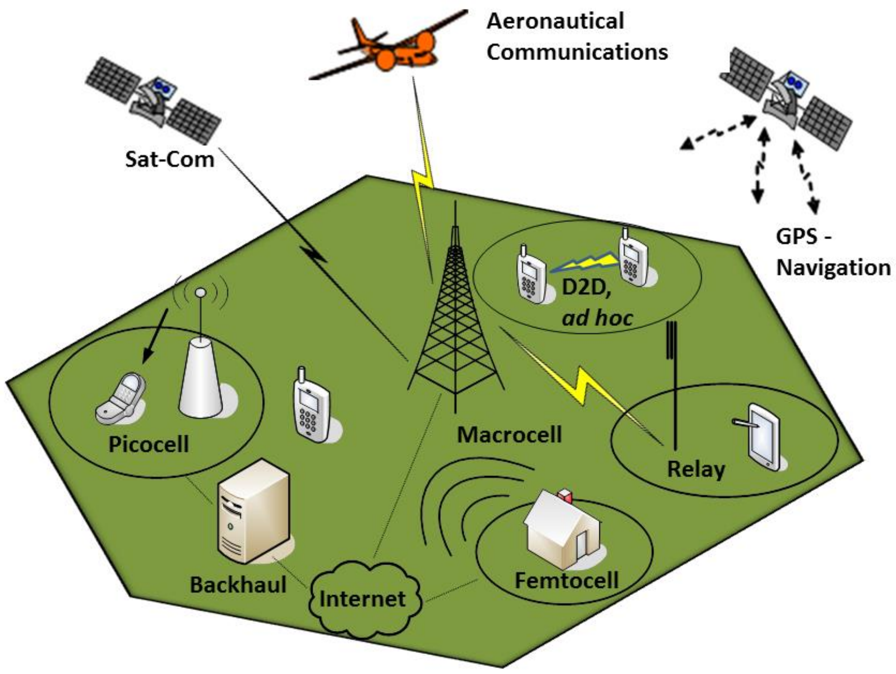
Vocabulary – Signals & Systems

- Signal
- System
- Continuous-Time
- Discrete-Time
- Impulse
- Step
- Transform
  - Time Domain
  - Frequency Domain
- Response
- Convolution

“Signal”




CARTOONSTOCK
.com
Reynolds
Search ID: dre2143
“Chief say he eat too many beans.”


Aeronautical
Communications
Sat-Com
GPS -
Navigation
D2D,
ad hoc
Picocell
Macrocell
Relay
Backhaul
Internet
Femtocell

"Signal"

A set of data or information!

“Signal”

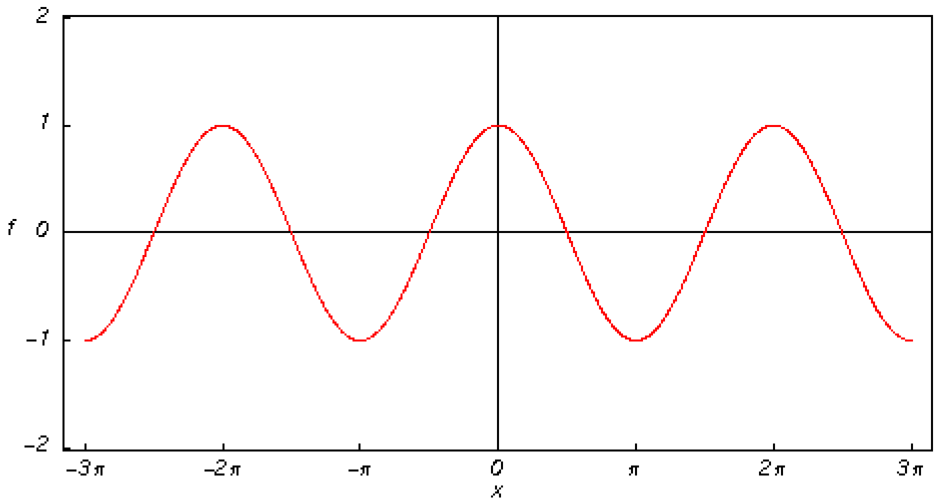
A set of data or information!

How do we normally represent data/information?

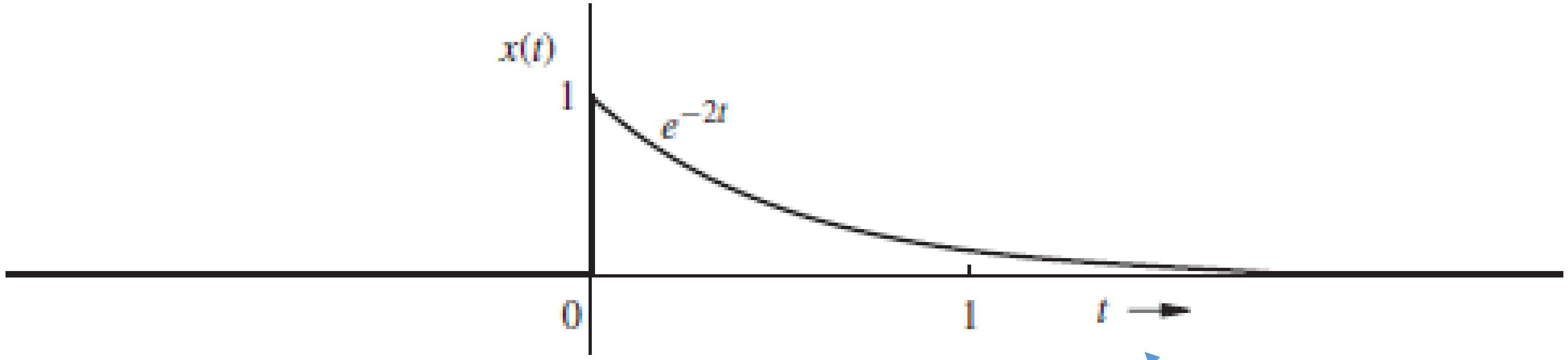
“Signal” A SET OF DATA OR INFORMATION!

.9998
.9994
.9986
.9976
.9962
.9945
.9925
.9903
.9877
.9848
.9816
.9781
.9744
.9703
.9659
.9613
.9563
.9511
.9455
.9397
.9336
.9272
.9205
.9135
.9063
.8988
.8910
.8829
.8746
.8660

$$\cos(t)$$



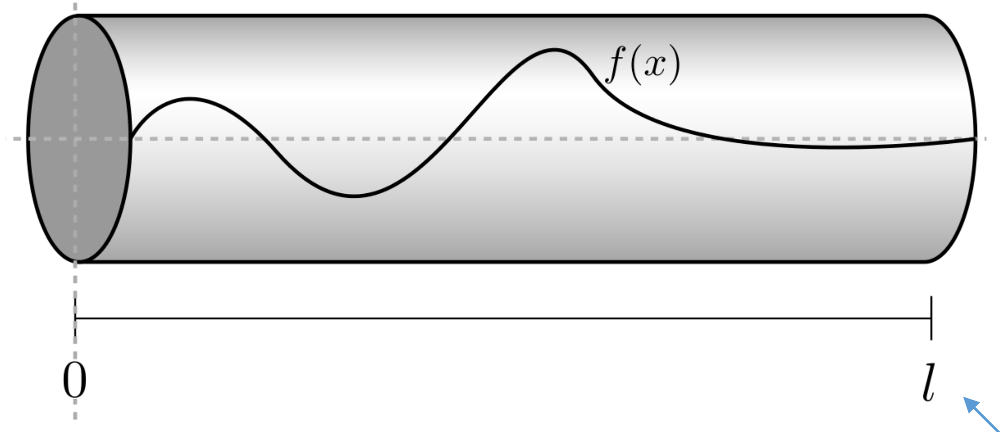
"Signal"



$$x(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

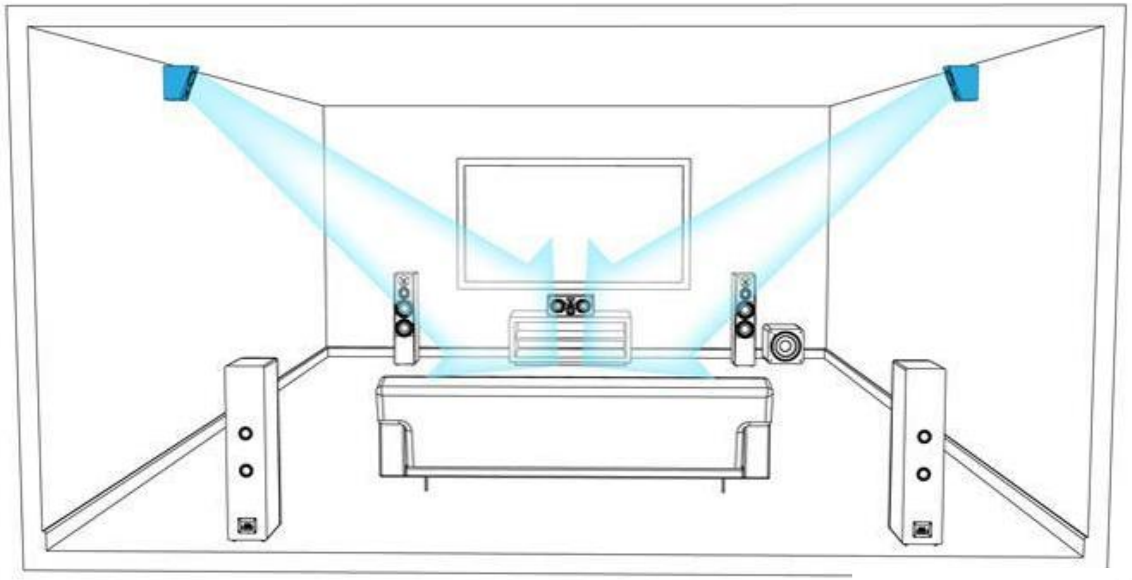
Our x-axis is often "time" (but doesn't have to be).

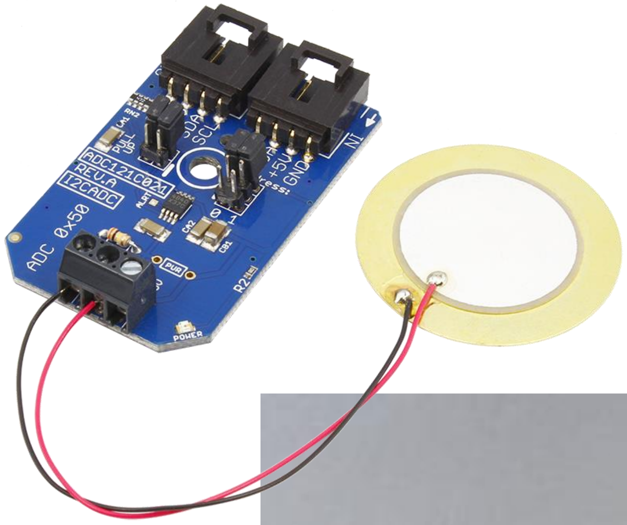
“Signal”

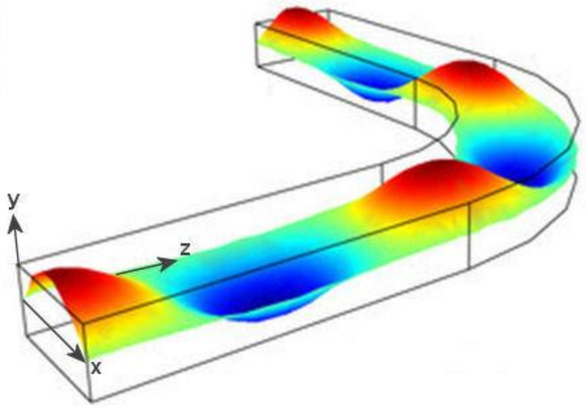


For temperature across a rod, x-axis could be along the length.

“System”







"System"

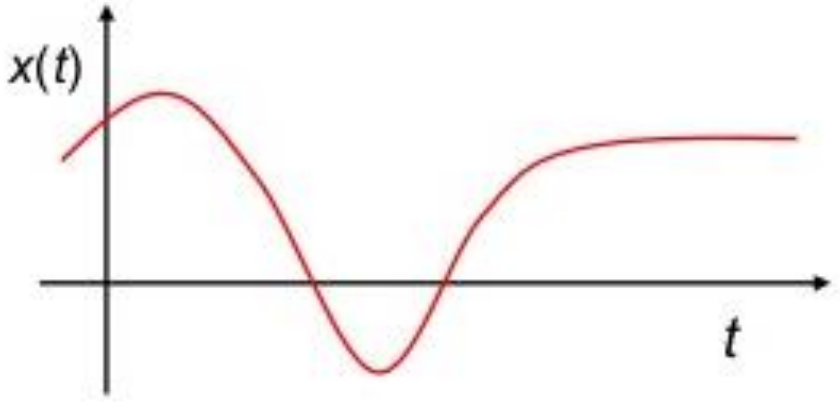
An entity that processes a signal

"System"

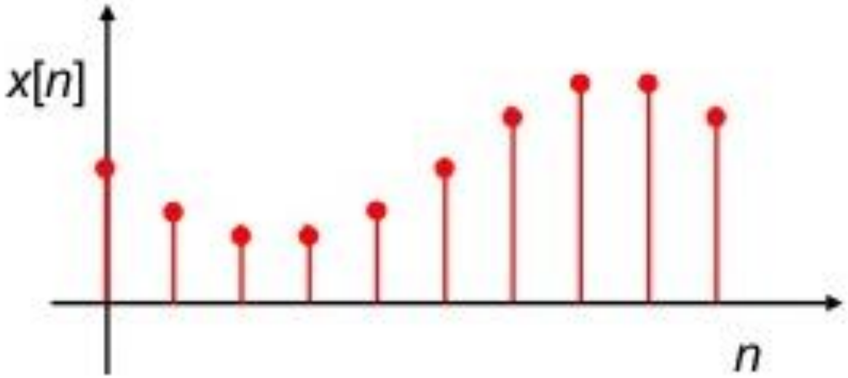
An entity that processes a signal

Can be something physical, or just an algorithm.

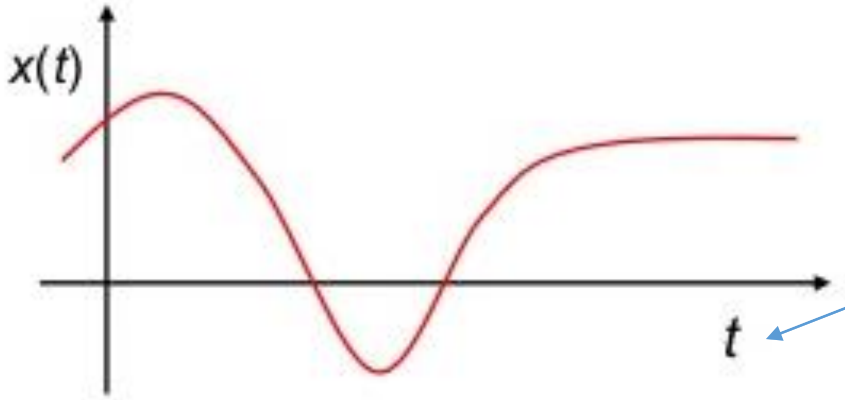
“Continuous-Time”



“Discrete-Time”

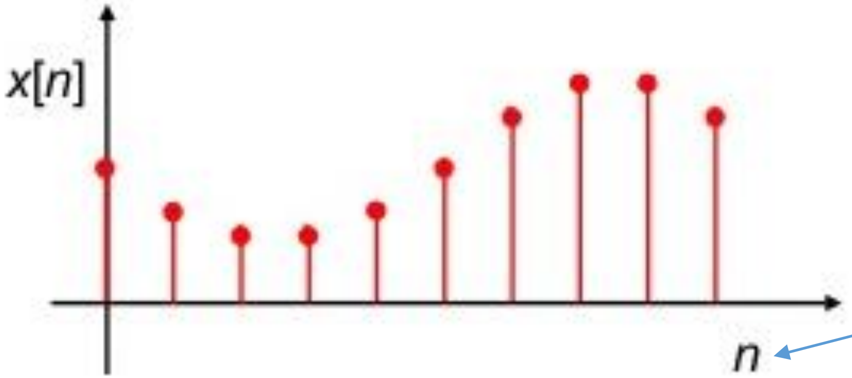


"Continuous-Time"



X-axis is a continuous variable (can take any value in a given range).

"Discrete-Time"



X-axis is a discrete variable (cannot take all values in a given range, e.g. $n = 1, 2, 3, \dots$)

“Impulse”

“Impulse”

impulse

noun

UK /ˈɪm.pʌls/ **US** /ˈɪm.pʌls/

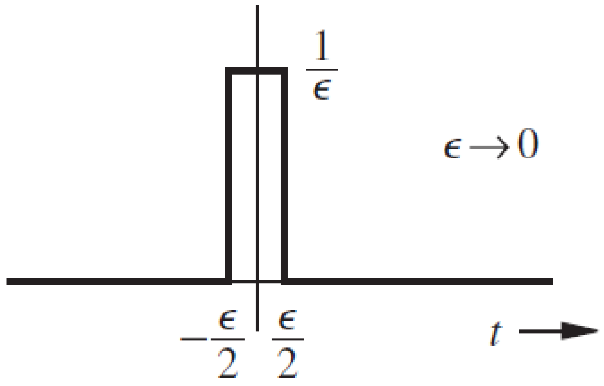
impulse ***noun*** **(WISH)**

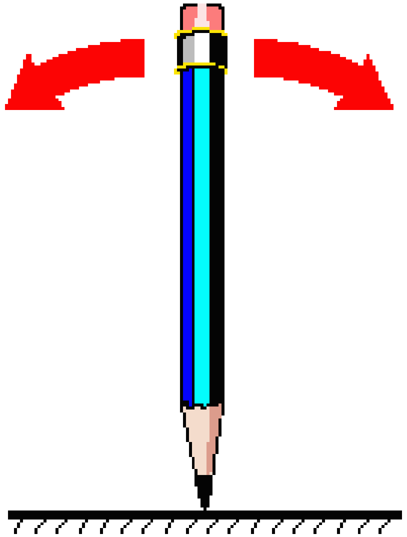
C2 [C + to infinitive]

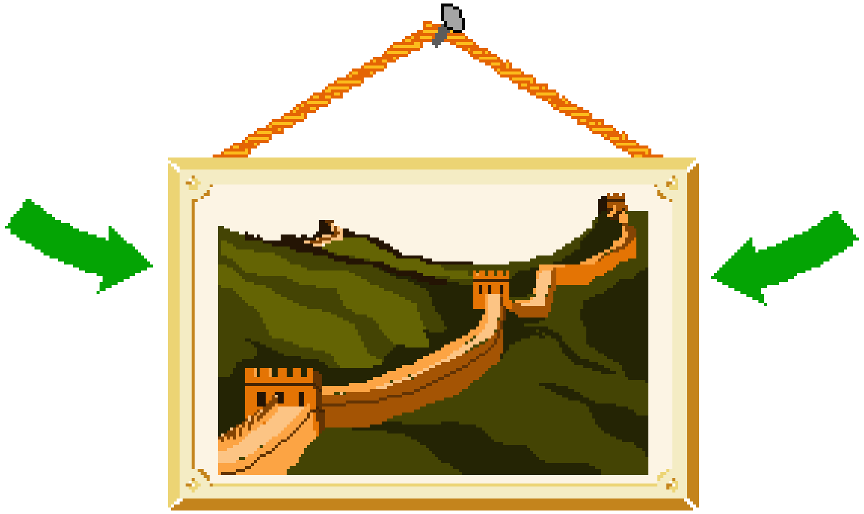
a sudden strong wish to do something:

“Impulse”

Sometimes we want to give a very **brief** “touch” to a system to see its reaction.

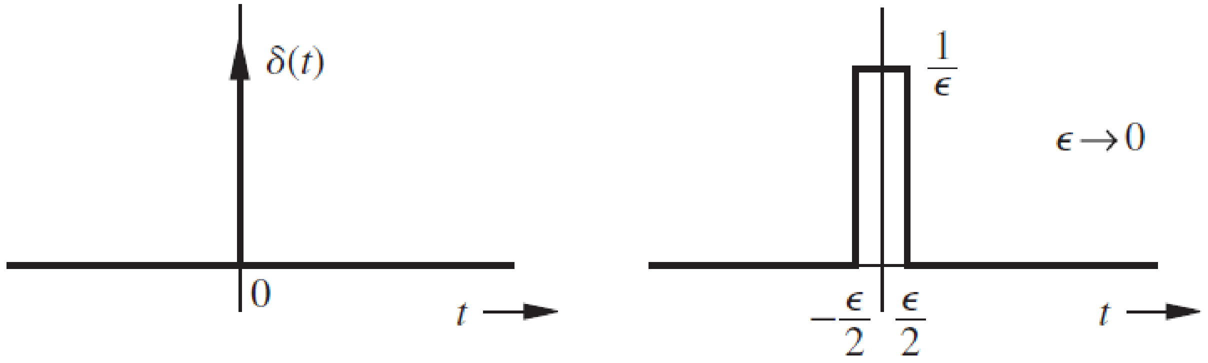






“Impulse”

Sometimes we want to give a very **brief** “touch” to a system to see its reaction.

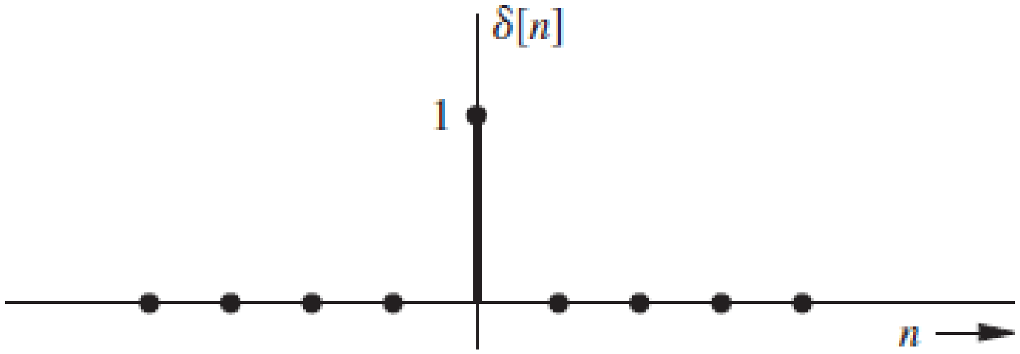


Continuous-Time Case

$$\delta(t) = 0 \quad t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

“Impulse”

Sometimes we want to give a very **brief** “touch” to a system to see its reaction.



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

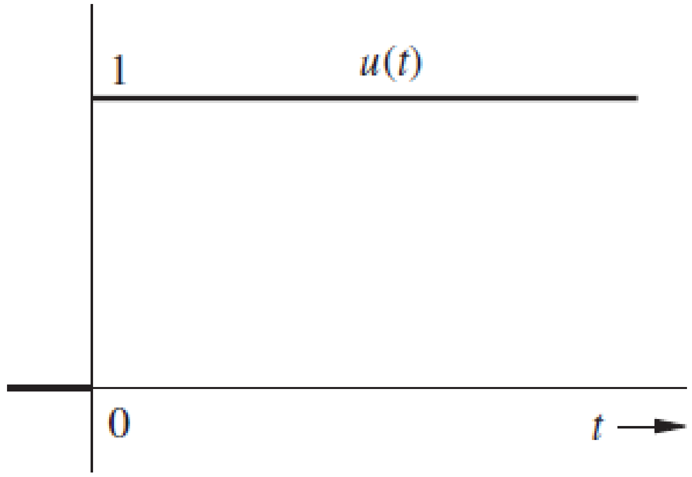
Discrete-Time Case

“Step”

Sometimes we want to give a **sustained** “push” to a system to see its reaction (or what happens).

“Step”

Sometimes we want to give a **sustained** “push” to a system to see its reaction.

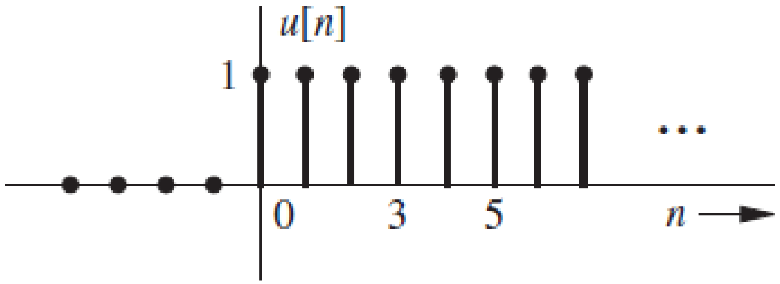


$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Continuous-Time Case

"Step"

Sometimes we want to give a **sustained** "push" to a system to see its reaction.



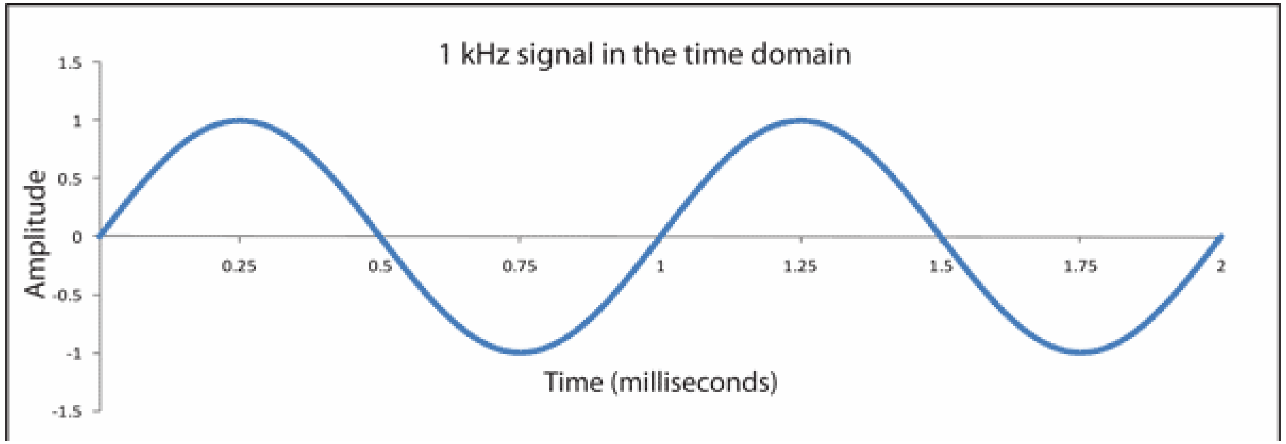
Discrete-Time Case

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

“Transform”

A transform is an **alternate form or representation** of something.

“Transform”

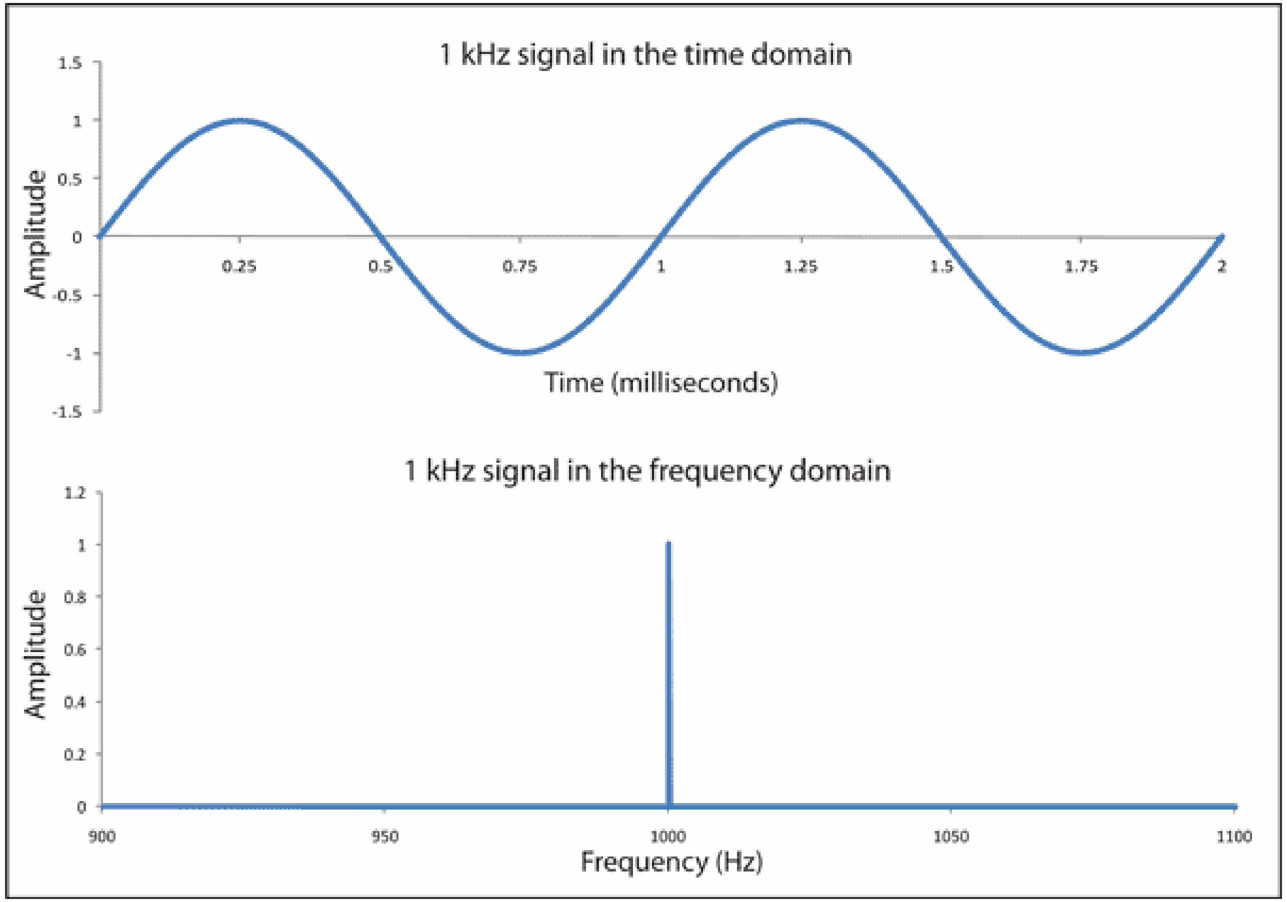


This is clearly a sine function. Only information I need to record is:

(a) This is a sine function
(b) It has amplitude 1
(c) It has frequency 1 kHz

"Transform"

A transform is an **alternate form or representation** of something.



Time Domain

Two alternate representations of $\sin(1000t)$

Frequency Domain

“Response”

The Case of Double Shah

Response is what a system does to a signal entering it.

Give me one rupee, and I’ll give you two!

“Response”

The Case of Double Shah

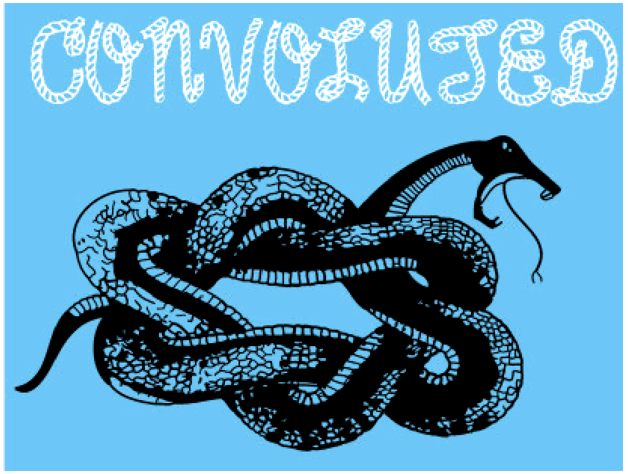
Response is what a system does to a signal entering it.

Give me one rupee, and I’ll give you two!

System “response”, let’s call it h, then $y = hx$ with $h = 2$.

“Convolution”

“Convolution”



convoluted

/ˌkɒnvəˈl(j)uːtɪd/

adjective

1. (especially of an argument, story, or sentence) extremely complex and difficult to follow.
   "the film is let down by a convoluted plot in which nothing really happens"

   Similar: complicated complex involved intricate elaborate impenetrable

2. TECHNICAL
   intricately folded, twisted, or coiled.
   "walnuts come in hard and convoluted shells"

“Convolution”

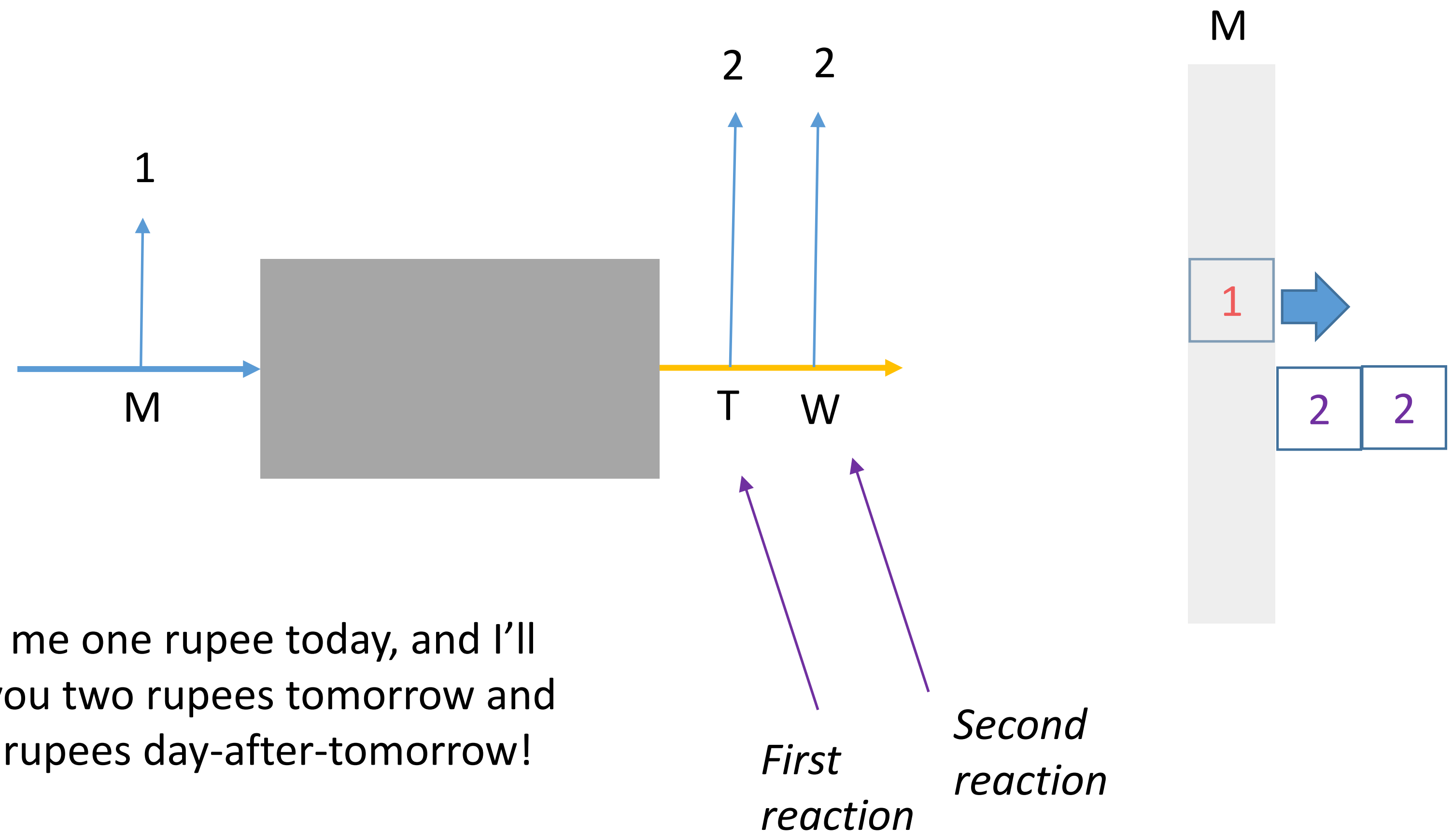
A mathematical operation (just as $+, -, \times, \div$) that helps us calculate the response of a special type of systems.

Convolution is often denoted by $*$

The Curious (and Completely Made Up) Case of Quadruple Shah

Give me one rupee today, and I'll
give you two rupees tomorrow and
two rupees day-after-tomorrow!

The Curious (and Completely Made Up) Case of Quadruple Shah



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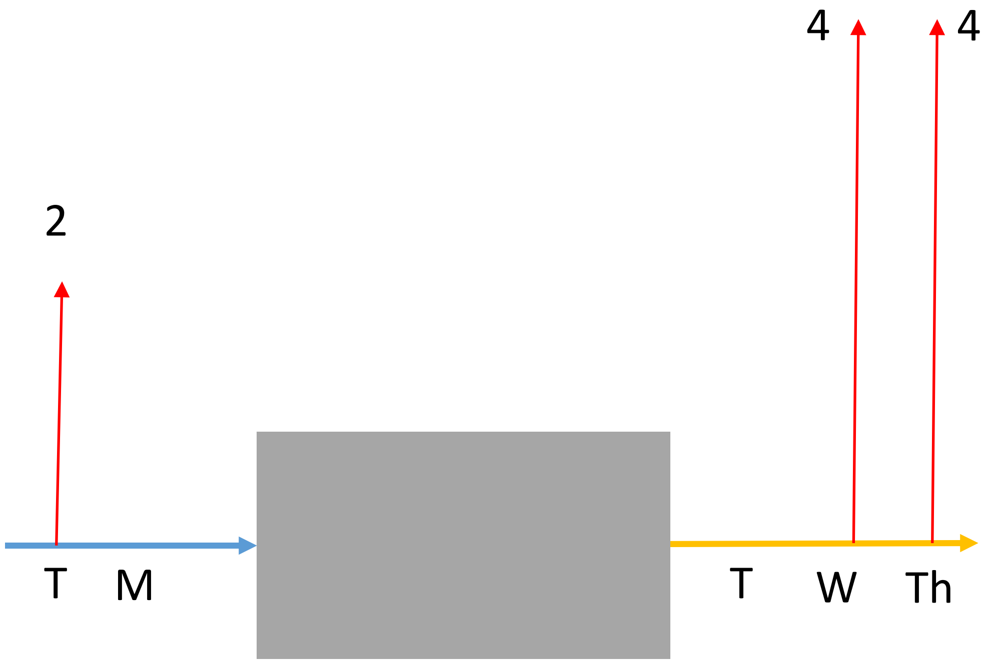

Give me one rupee today, and I'll
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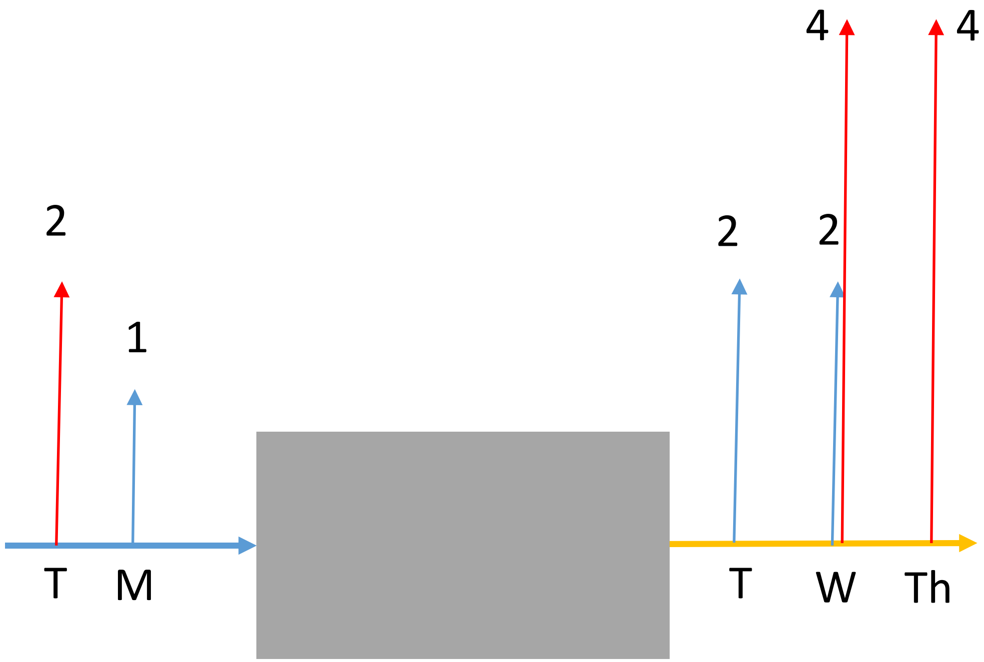
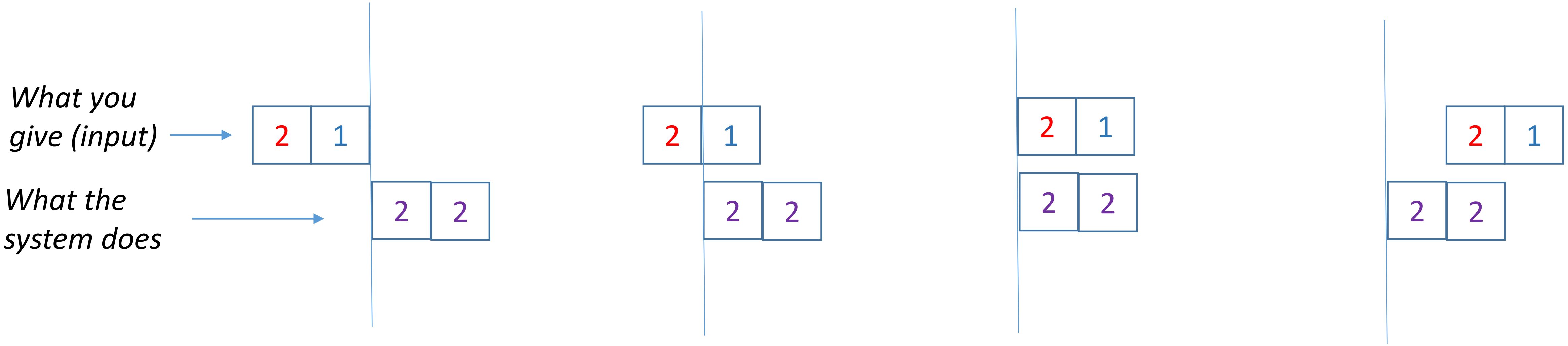


Tuesday	2
Wednesday	2+4 = 6
Thursday	4

Convolution helps us calculate such outputs mathematically.

Convolution Shortcut: **Multiply overlapping cells and add the results! (then slide input right and repeat!)**



On Monday you get nothing!

On Tuesday you get $2 \times 1 = 2$ rupees

On Wednesday you get $2 \times 2 + 2 \times 1 = 6$ rupees

On Thursday you get $2 \times 2 = 4$ rupees

We will see a precise formulation of convolution later in the course…

Questions?? Thoughts??

ES 332
Signals and Systems

with

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Signal Basics I

Types

Several classifications of signals.

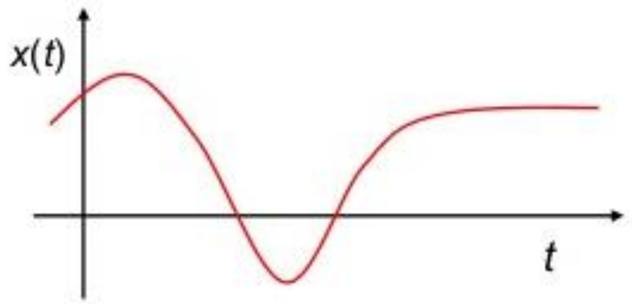
Examples

Some practice problems

Engineers like to classify…

1. Is the x-axis continuous or discrete?

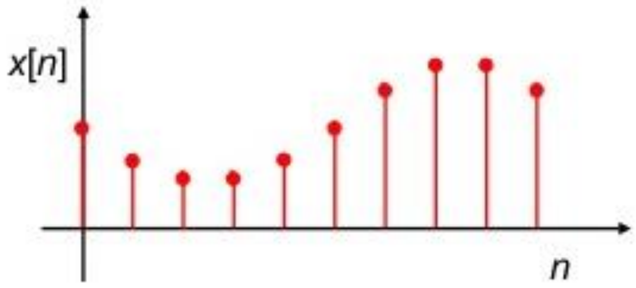
"Continuous-Time"



X-axis is a continuous variable (can take any value in a given range).

Vs.

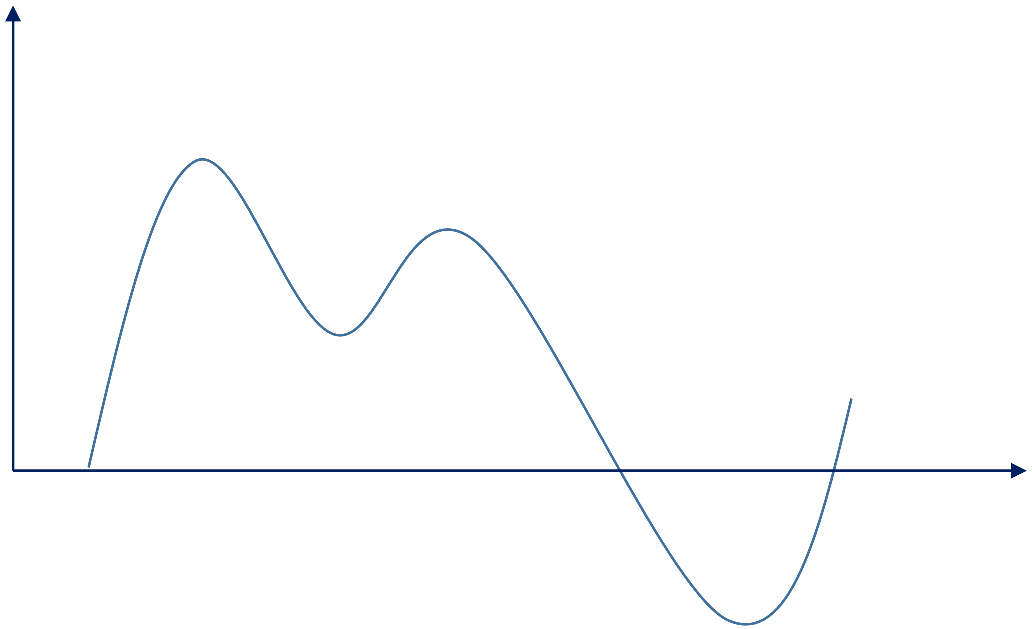
"Discrete-Time"

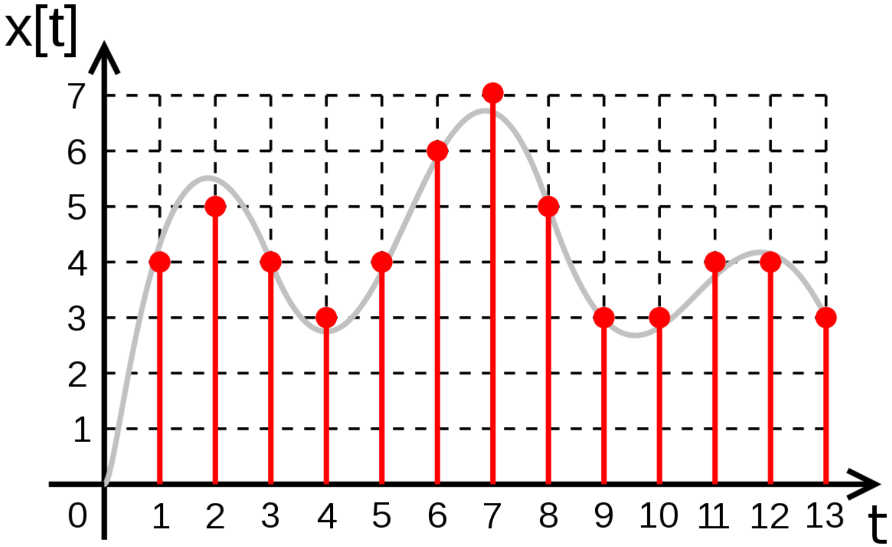


X-axis is a discrete variable (cannot take all values in a given range, e.g. $n = 1, 2, 3, \ldots$)

2. Is the y-axis also discrete (or continuous)?

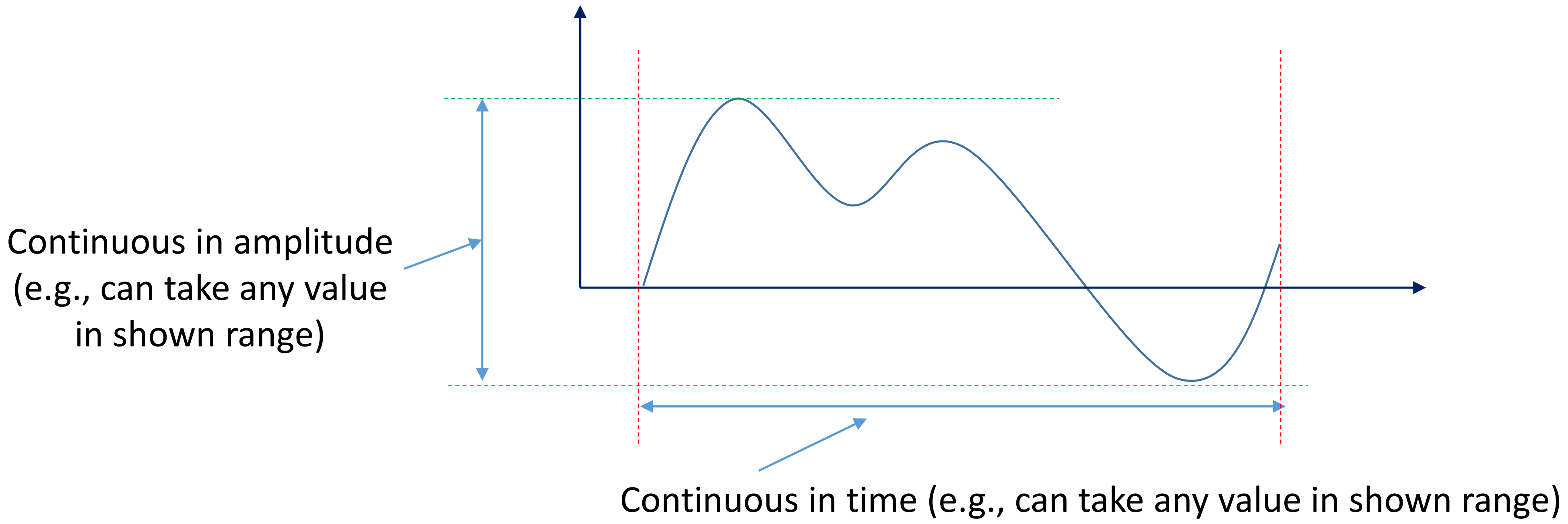
"Analog" Vs. "Digital"





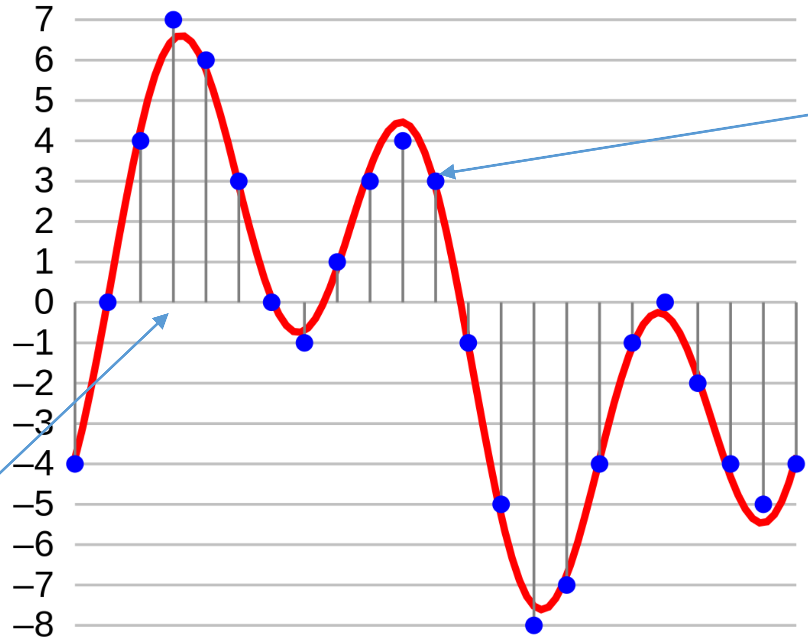
Analog

- Most signals in real life are analog.
  - Analog = both x and y axes are continuous.
  - Continuous? No breaks. Can take **any** value in a given range.
  - e.g., temperature in this room



Digital

- We often digitize analog signal
  - Digitize? = make both x and y axes discrete.



discontinuous in time (e.g., values taken only at fixed times but not in-between)

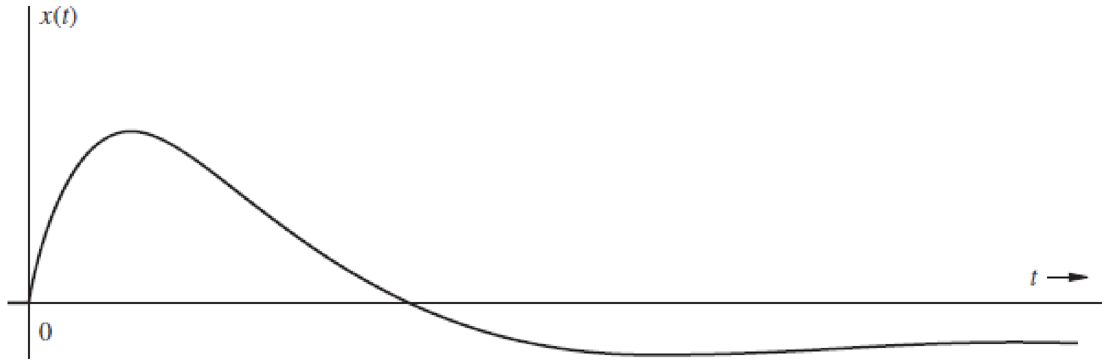
“discontinuous” in amplitude (e.g., can not take values between 3 and 4, like 3.5 etc.)

3.Does the signal have a repeating pattern?

"Periodic"

Vs.

"Aperiodic"



Periodic Signal’s Time Period & Frequency

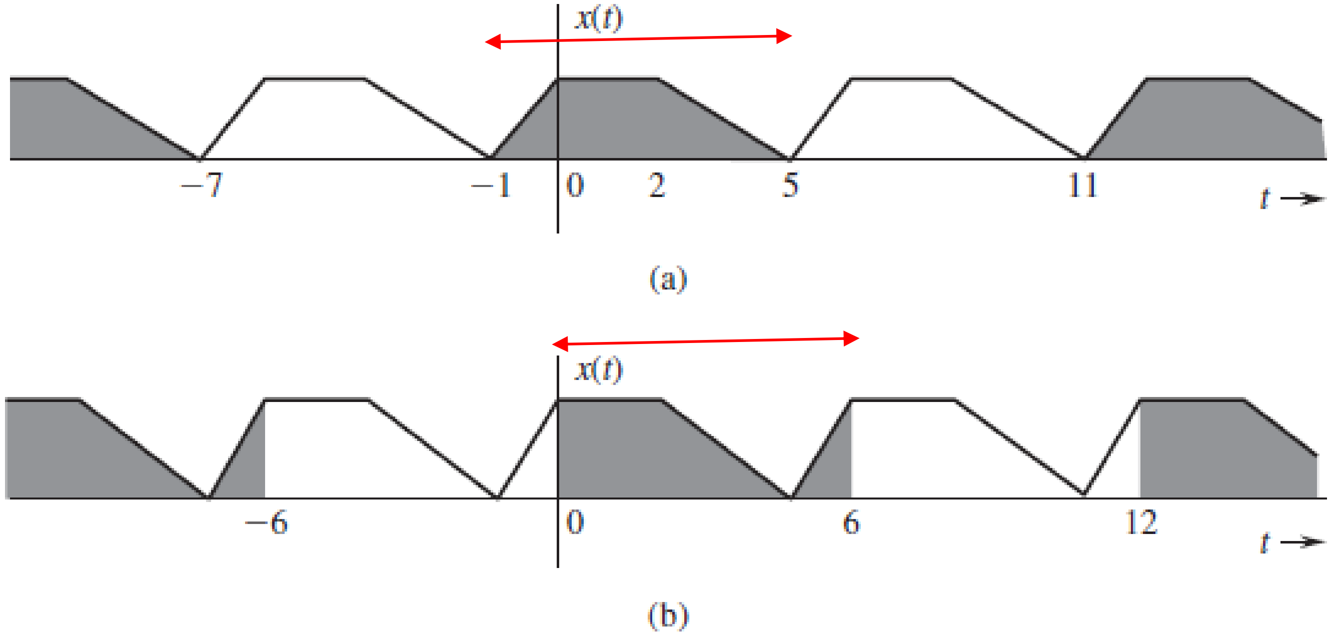


Figure 1.13 Generation of a periodic signal by periodic extension of its segment of one-period duration.

T_0 = time period = length of the minimum repeating pattern.

$f_0 = \frac{1}{T_0}$ = fundamental frequency

$$\boxed{x(t) = x(t + T_0) \qquad \text{for all } t}$$

Periodic Signal's Time Period & Frequency

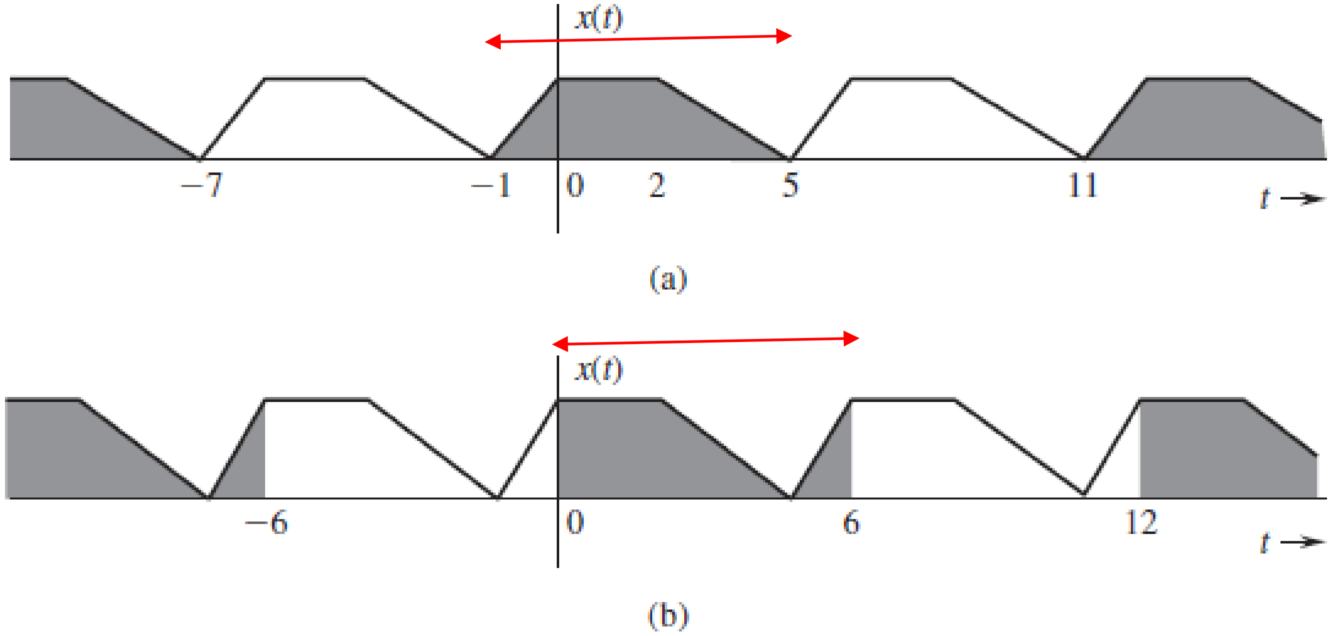


Figure 1.13 Generation of a periodic signal by periodic extension of its segment of one-period duration.

$$\boxed{x(t) = x(t + T_0) \qquad \text{for all } t}$$

e.g., if $T_0 = 6$, then

$$x(0) = x(0 + 6)$$
$$x(1) = x(1 + 6)$$
$$x(-1) = x(-1 + 6)$$

Periodic Signal : Area Under One Period

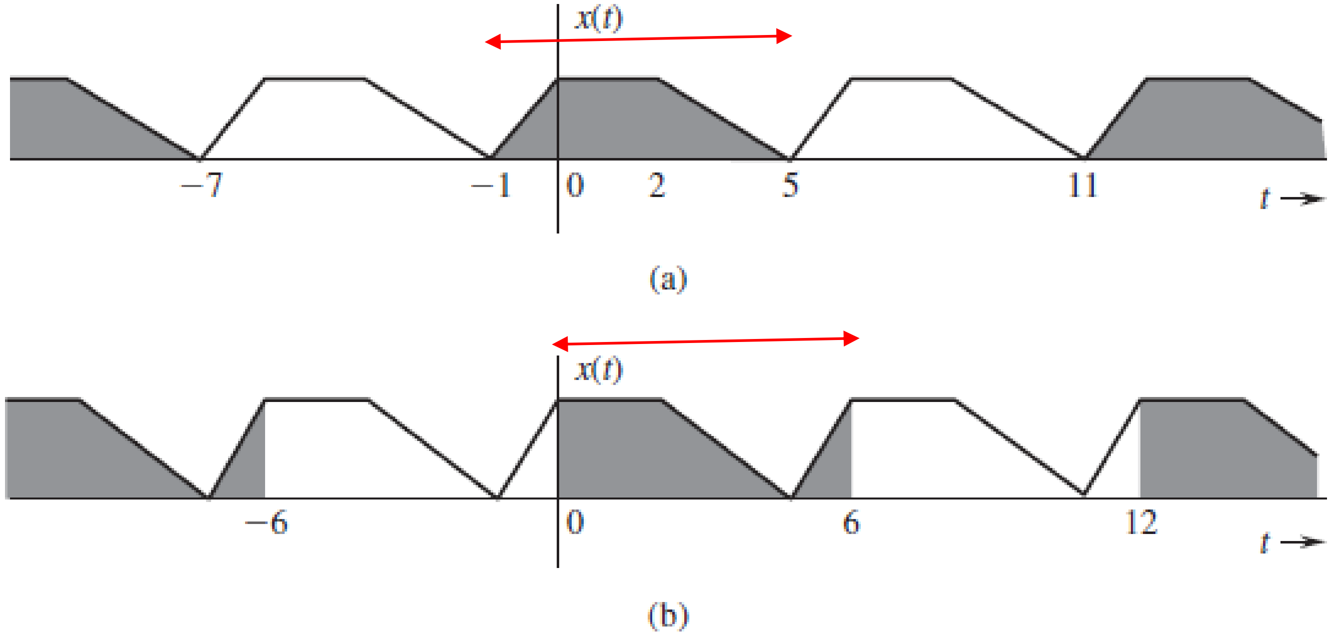


Figure 1.13 Generation of a periodic signal by periodic extension of its segment of one-period duration.

$$\int_a^{a+T_0} x(t)\,dt = \int_b^{b+T_0} x(t)\,dt$$

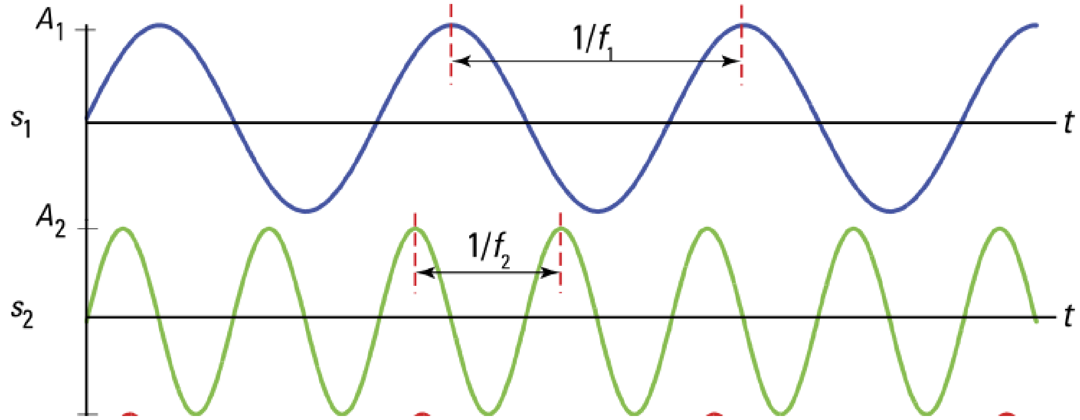
Area under one whole period of a periodic signal is always the same no matter where you start!!

3. Are the Signal Values/Parameters Random?

"Deterministic"

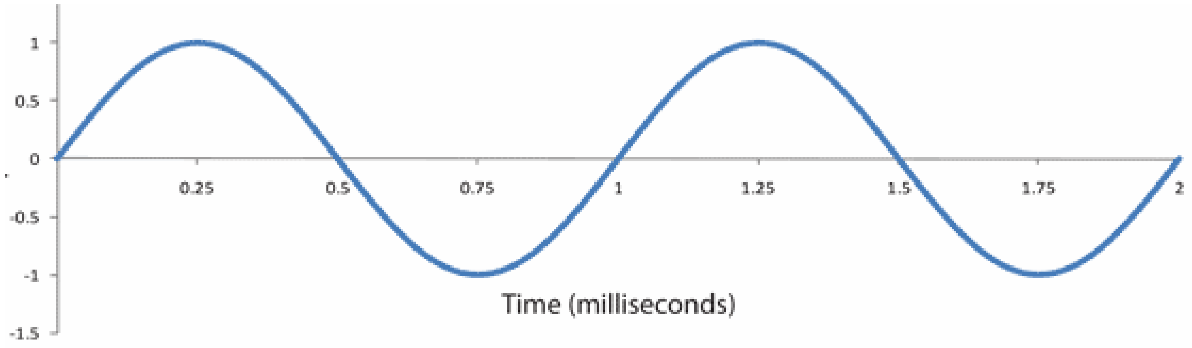
Vs.

"Random"



3. Are the Signal Values/Parameters Random?

"Deterministic"



$$x(t) = \sin(2\pi f t) \text{ with } f = 1\,kHz$$

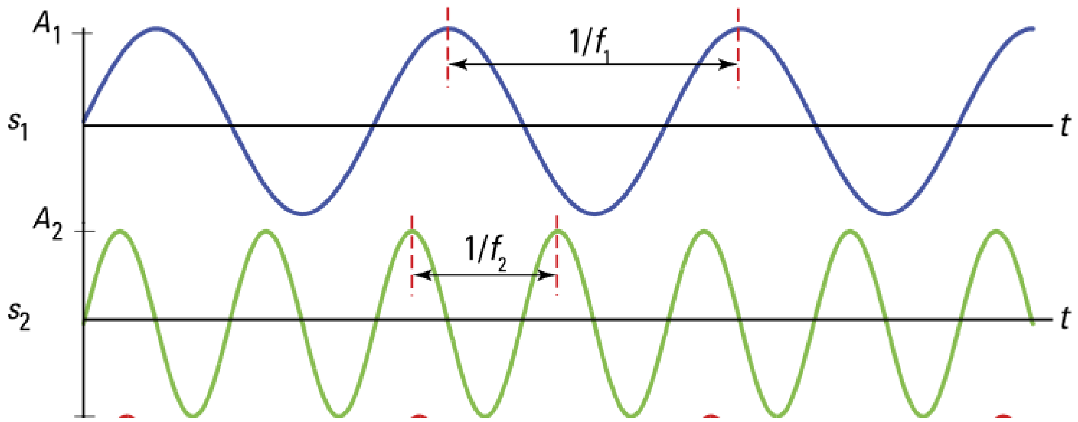
3. Are the Signal Values/Parameters Random?

"Random"

$$x(t) = \sin(2\pi f t)$$

Value of f to be decided by tossing a coin

$$f = \begin{cases} 1, & Heads \\ 2, & Tails \end{cases}$$



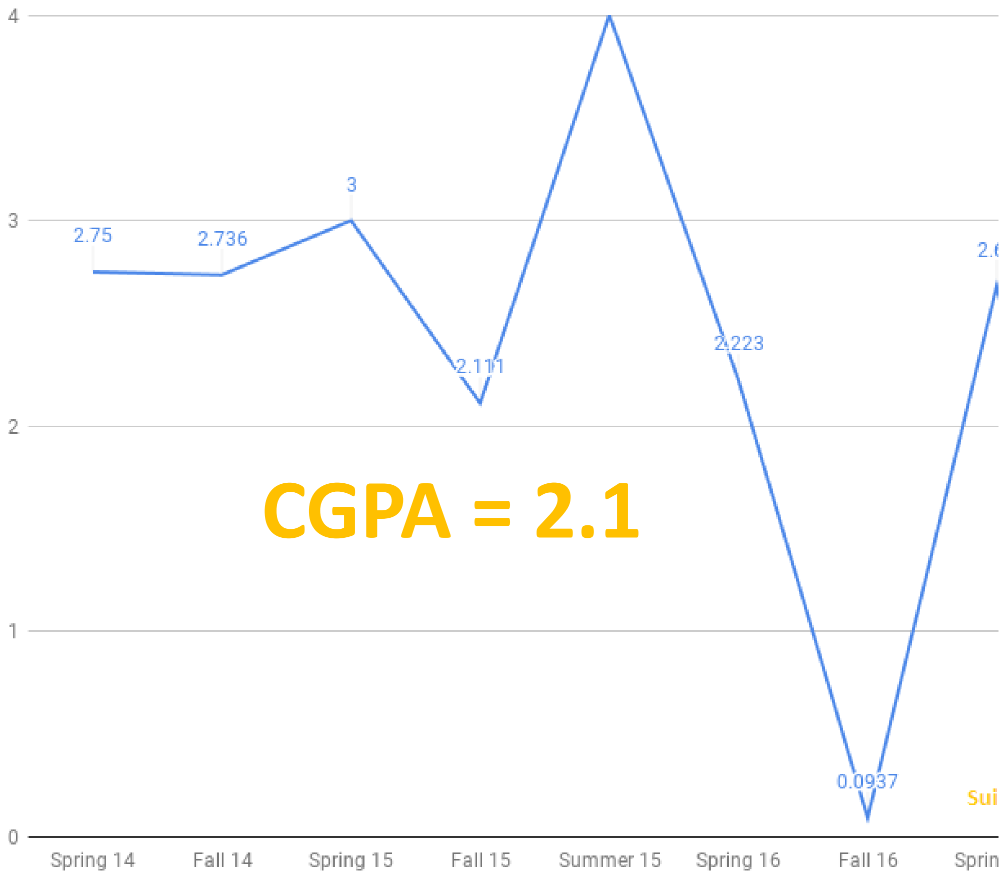
4. Does the signal have finite energy or finite Power?

- What are energy and power of a signal?
- Why are they needed?

Measures are important…

Height, Age, GDP, Stock Index…

How can we measure a signal?



We often find it useful to look at just one value that gives the overall effect of a signal.

For example, in place of looking at your semester-wise GPA, employer may look at your CGPA.

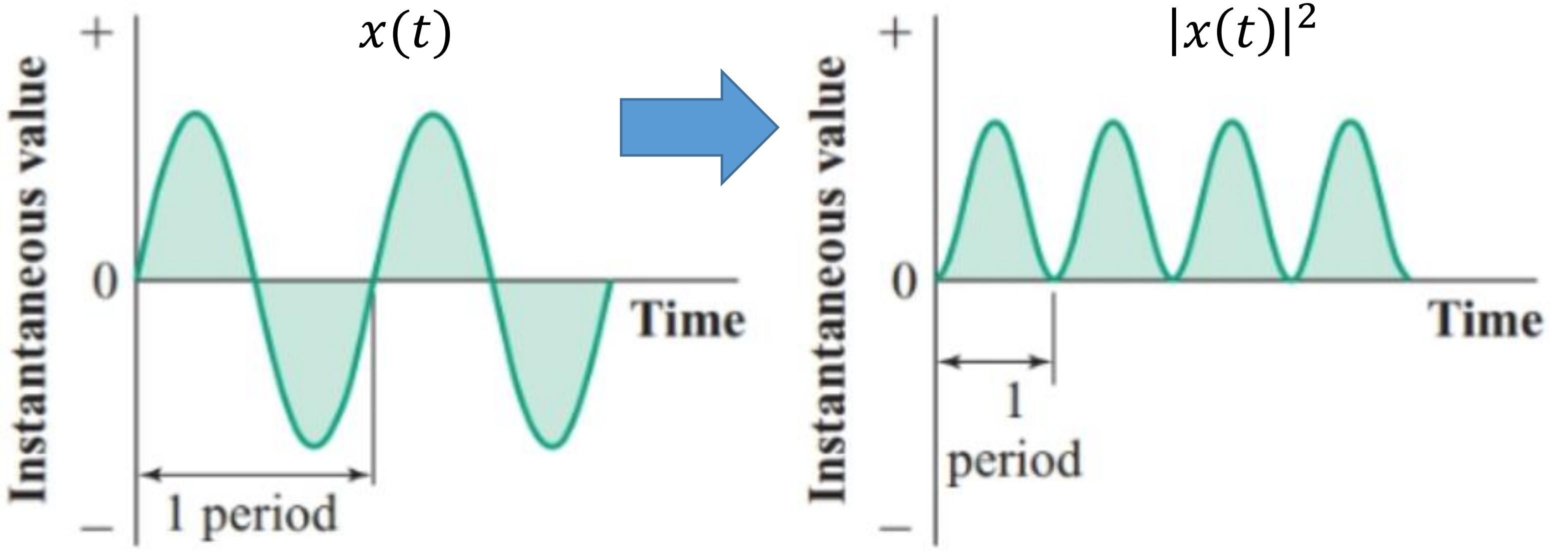
Two Common Measures of a Signal: Energy & Power

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Signal Energy is the area under the absolute square of the signal.

Signal Energy

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

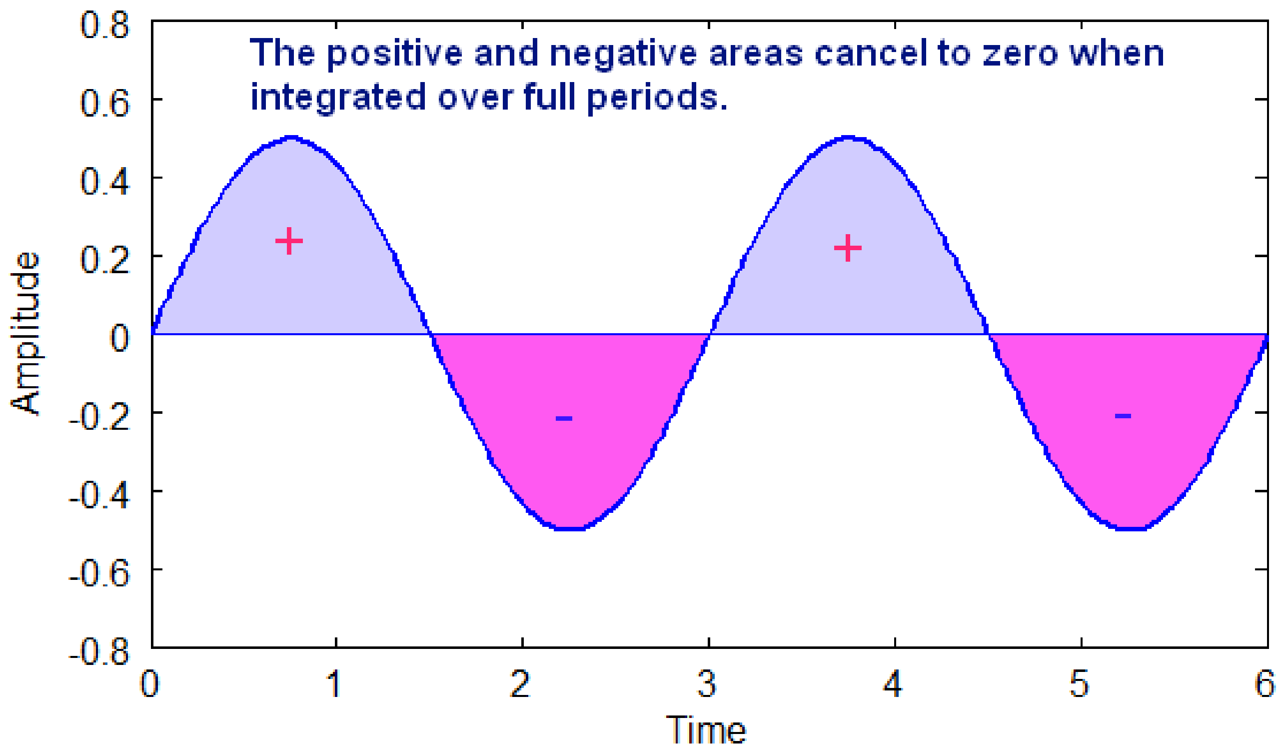


(a) Sine wave

(b) Sine-squared wave

Signal Energy: why don't we just use $x(t)$?

Signal Energy: why don't we just use $x(t)$?



For many signals, area under the signal curve will turn out to be zero. So not a good way of measuring signal.

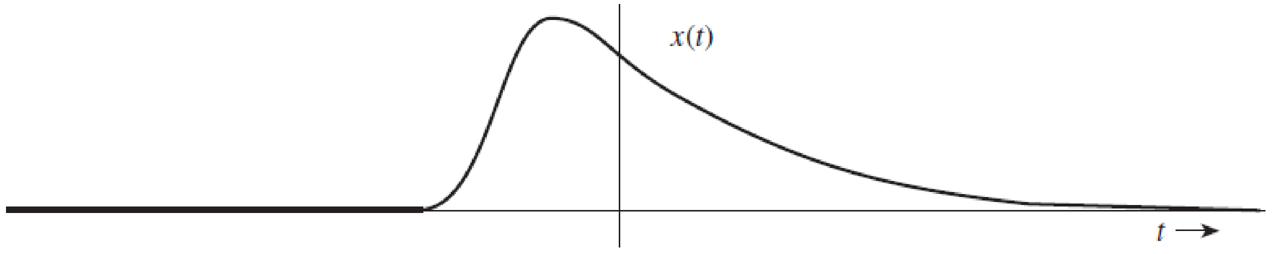
$$\int_{-\infty}^{\infty} x(t)dx = 0$$

Energy Signal = A signal that has finite energy

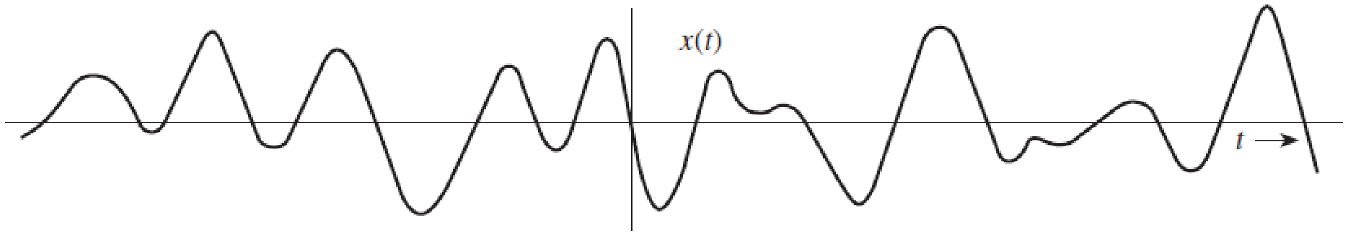
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \neq \infty$$

Energy Signal = A signal that has finite energy

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \quad \neq \infty$$



(a)



(b)

Which one has finite energy?

Many theoretical signals do not have a finite energy…

In that case it is more useful to measure their **average energy per unit time** (called "Power")

Many theoretical signals do not have a finite energy…

In that case it is more useful to measure their **average energy per unit time** (called "Power")

$$\boxed{P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt}$$

Power Signal = A signal whose power is neither infinite nor zero.

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \quad \neq \infty$$

and

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \quad \neq 0$$

Energy & Power – Some interesting facts…

- There are signals that are neither energy signals nor power signals
- An energy signal can never be a power signal
- A power signal can never be an energy signal
- All practical (real-life) signals are energy signals
- Periodic signals are often power signals

For a periodic signal the power formula can be simplified to:

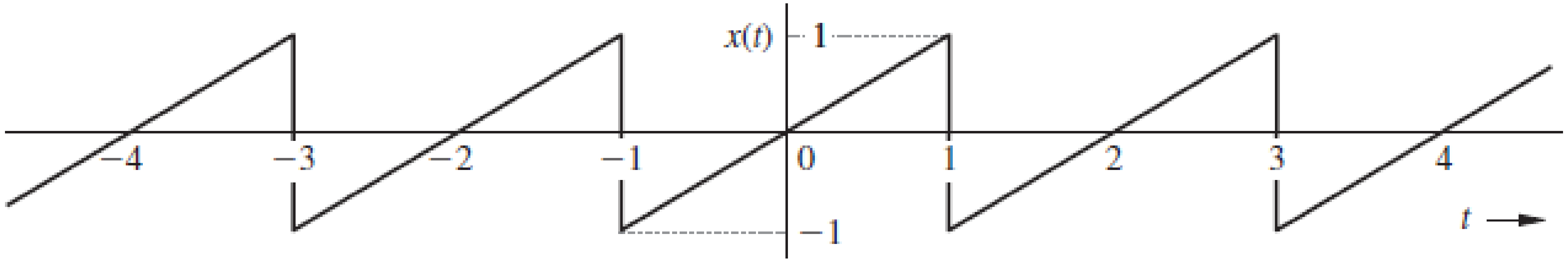
$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt$$

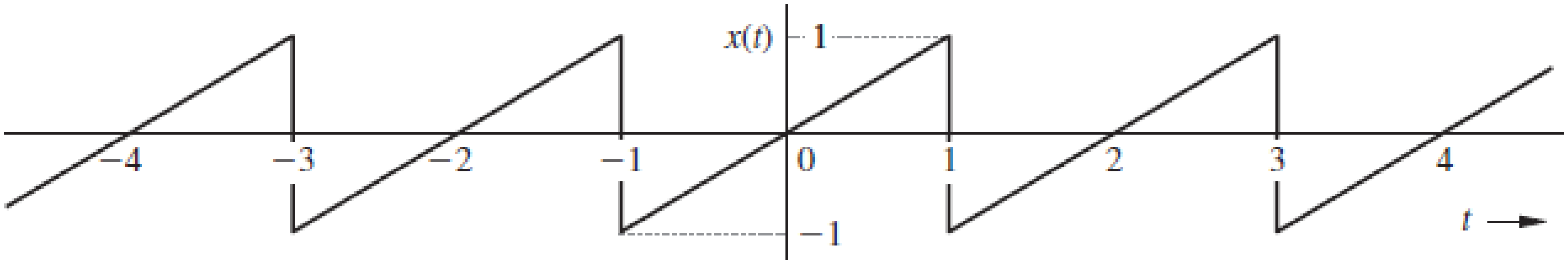
i.e., integrate over one period and divide by the period (T_0).

RMS – root-mean-squared value

$$rms = \sqrt{P_x}$$

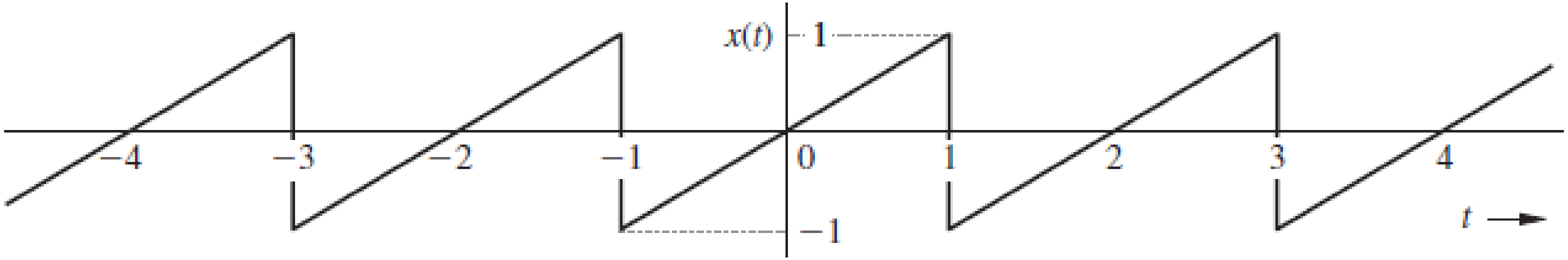
Examples


x(t)
1
−4
−3
−2
−1
0
1
2
3
4
−1
t



First we note that

- The signal is infinitely long, and not decaying
- It is periodic with period $T_0 = 2$
- Its period from -1 to 1 can be modeled as $x(t) = t$

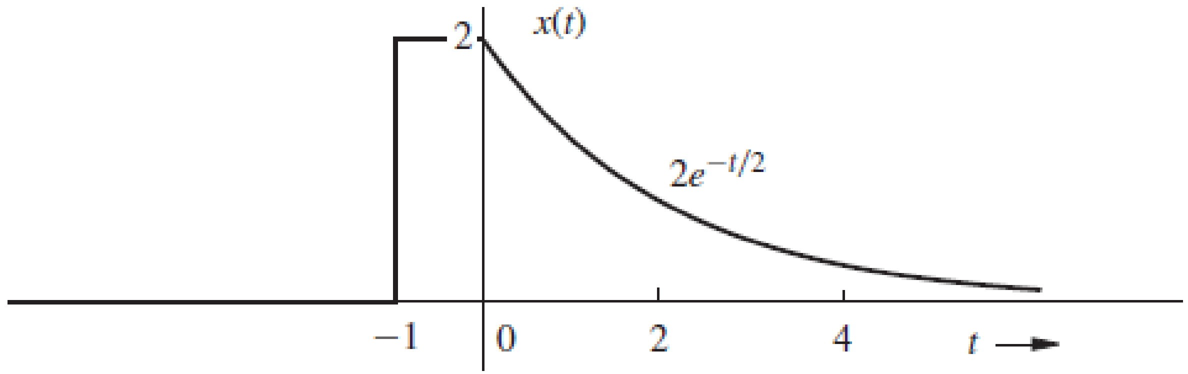


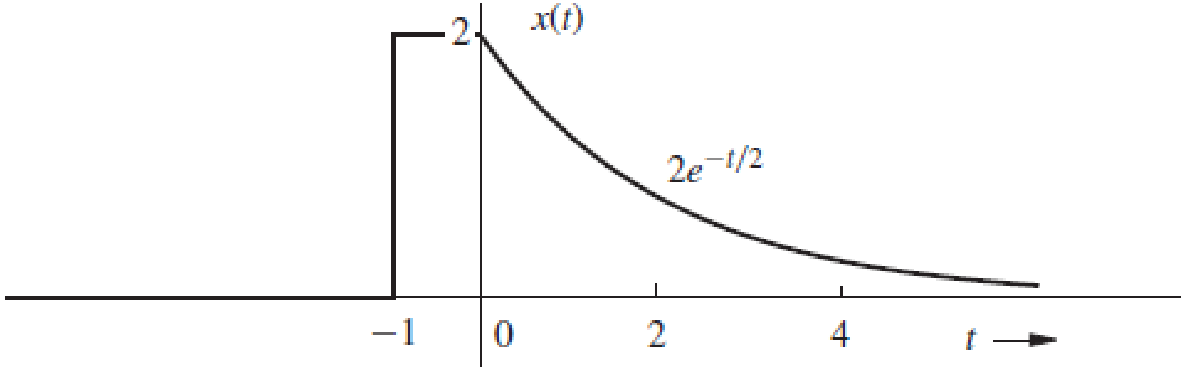
$$E_x = \infty$$

$$P_x = \frac{1}{2}\int_{-1}^{1} |x(t)|^2\, dt = \frac{1}{2}\int_{-1}^{1} t^2\, dt = \frac{1}{3}$$

$$rms = \sqrt{P_x} = \frac{1}{\sqrt{3}}$$

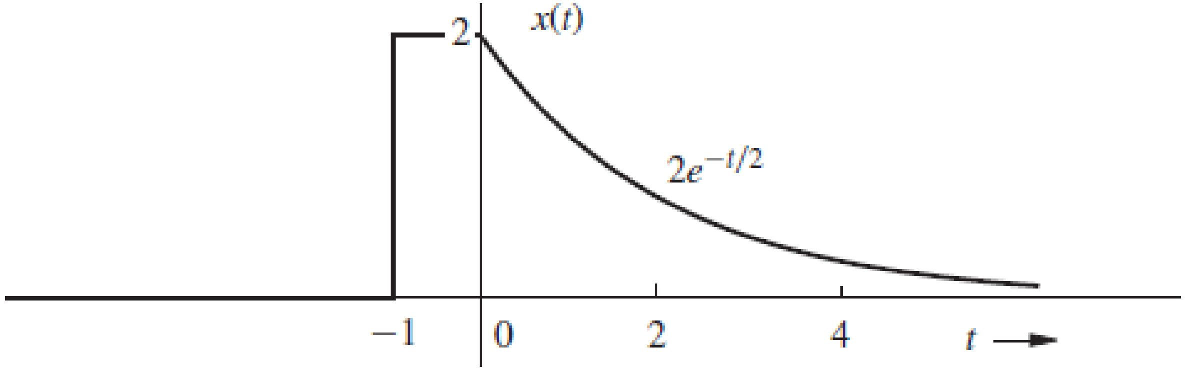
Since power is finite and non-zero, we conclude that this is a Power Signal.


2
x(t)
2e^{−t/2}
−1
0
2
4
t



First we note that

- The signal is infinitely long, but asymptotically decaying
- It is not periodic
- It can be modeled as
  - $x(t) = \begin{cases} 2, & -1 \leq t \leq 0 \\ 2e^{-t/2}, & t \geq 0 \end{cases}$



$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-1}^{0} (2)^2\,dt + \int_{0}^{\infty} 4e^{-t}\,dt = 4 + 4 = 8$$

- Since energy is finite, this is an Energy Signal
- Since energy signals cannot be power signals, this is not a power signal.

Show that $\cos(\omega_0 t)$ is periodic, with period $T_0 = \dfrac{2\pi}{\omega_0}$

For periodicity, we must have (for all t):

$$\cos(\omega_0 t) = \cos\big(\omega_0 (t + T_0)\big)$$

$$\boxed{x(t) = x(t + T_0) \qquad \text{for all } t}$$

$$RHS = \cos\left(\omega_0\left(t + \frac{2\pi}{\omega_0}\right)\right) = \cos(\omega_0 t + 2\pi) = \cos(\omega_0 t) = LHS$$

Since $\cos\theta$ is periodic, with period 2π

Questions?? Thoughts??

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Signal Basics II

Operations

Messing with signals

Models

Common signals

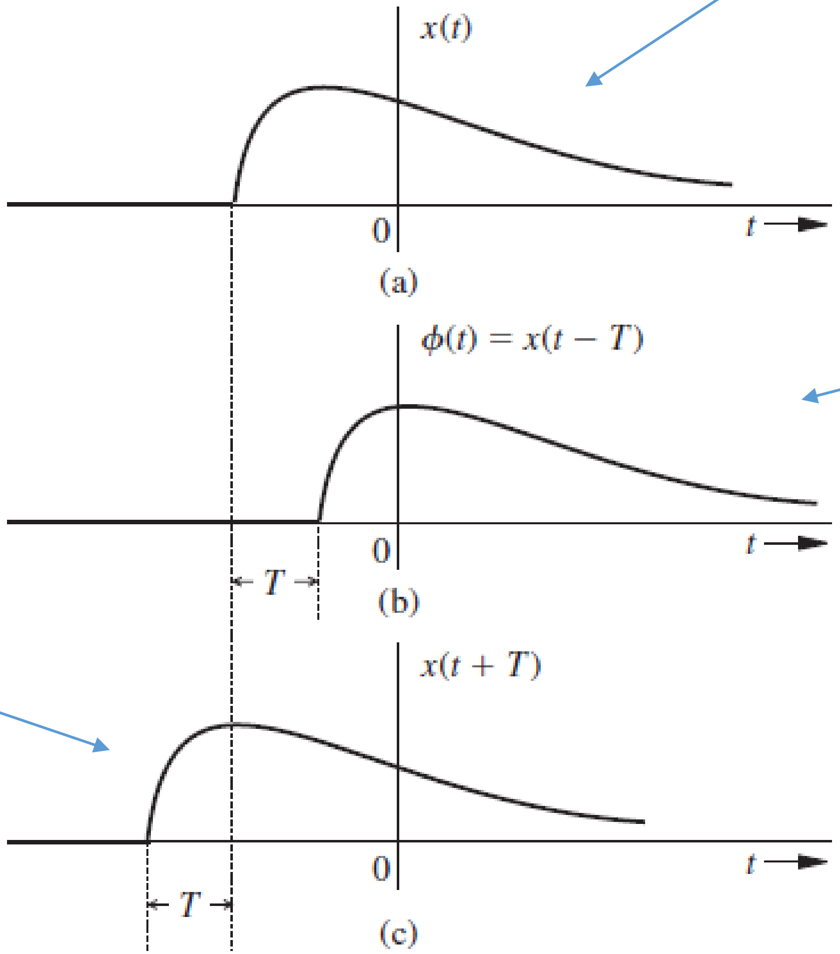
Examples

Some practice problems

Systems often alter signals. It is good to know some of these changes…

1. Time-Shifting

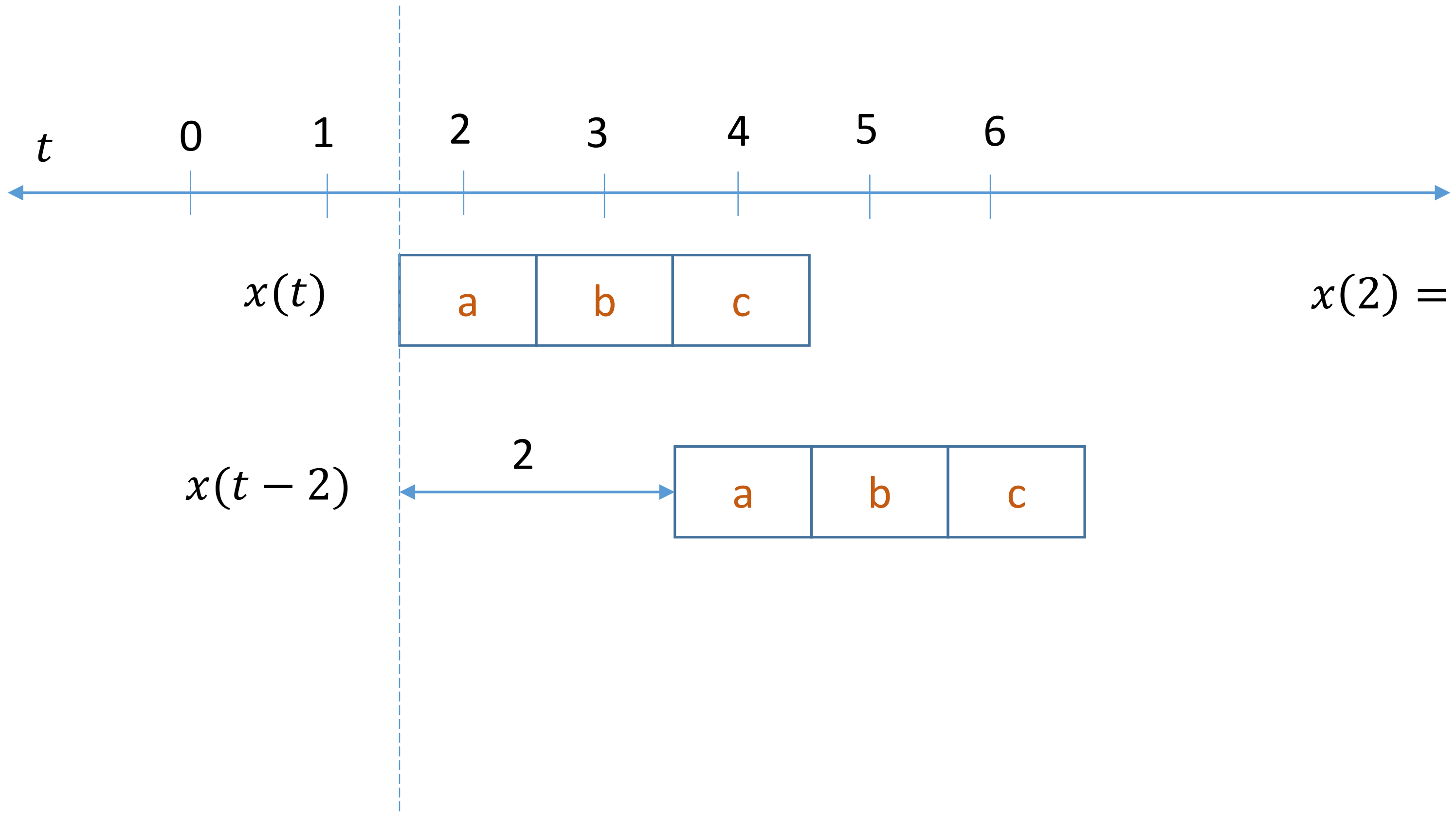
Original signal



Right-shifted (delayed), i.e. **occurs later than the original**.

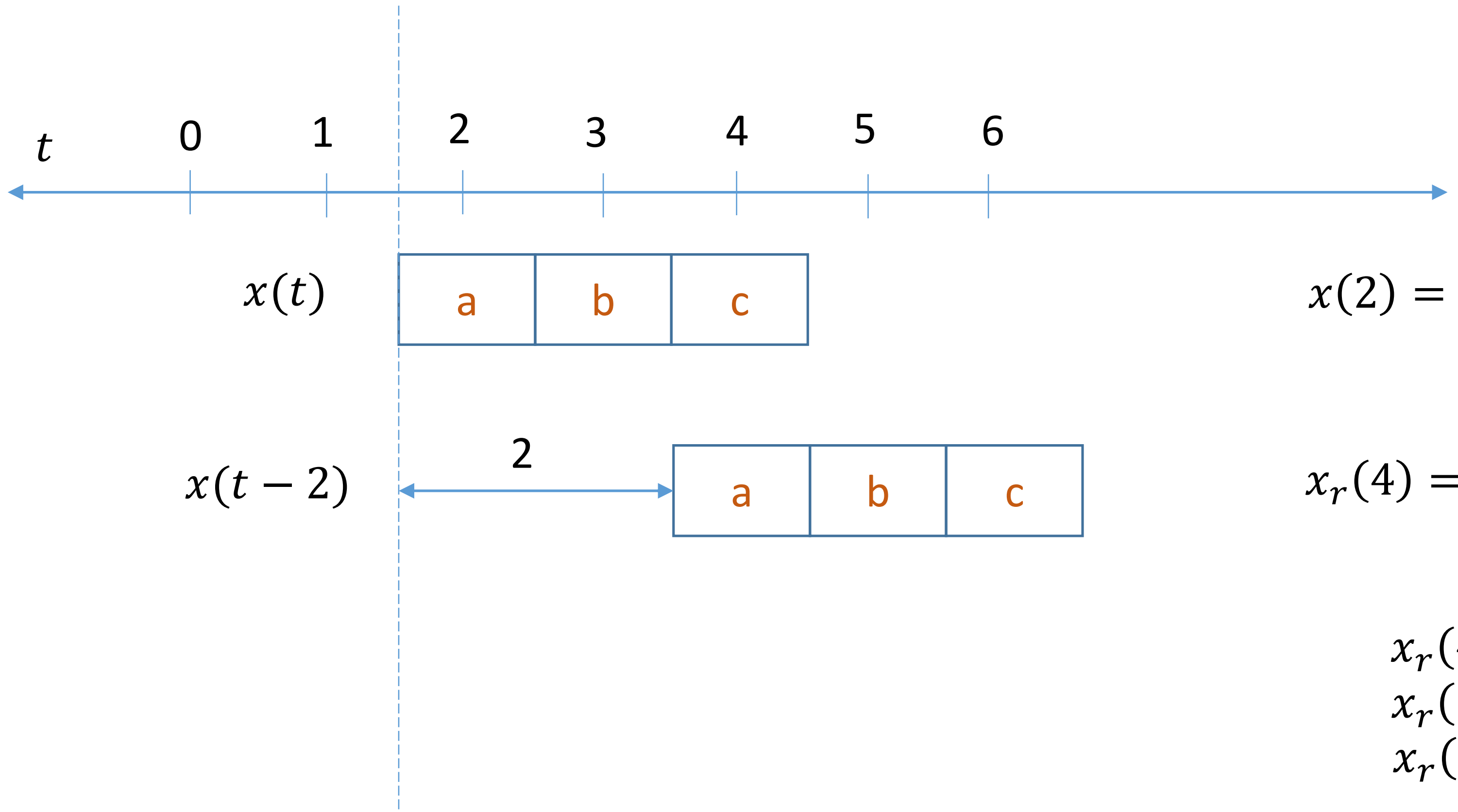
Left-shifted (advanced), i.e. **occurs earlier than the original**.

1. Time-Shifting: How to write mathematically?



$x(2) = a, x(3) = b, x(4) = c$

1. Time-Shifting: How to write mathematically?

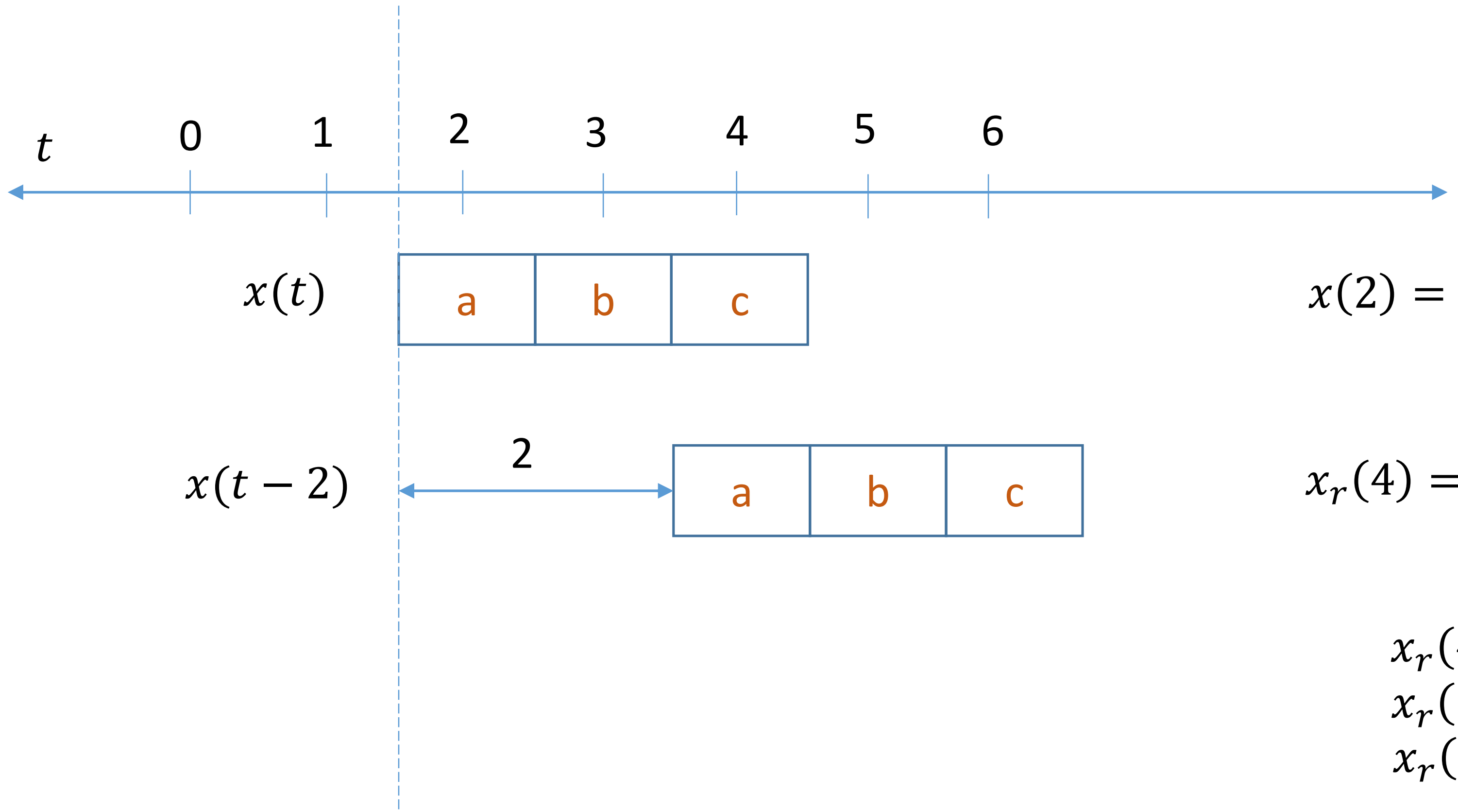


$x(2) = a, x(3) = b, x(4) = c$

$x_r(4) = a, x_r(5) = b, x_r(6) = c$

$$x_r(4) = x(2) = a$$
$$x_r(5) = x(3) = b$$
$$x_r(6) = x(4) = c$$

1. Time-Shifting: How to write mathematically?



$x(2) = a, x(3) = b, x(4) = c$

$x_r(4) = a, x_r(5) = b, x_r(6) = c$

$$x_r(4) = x(2) = a$$
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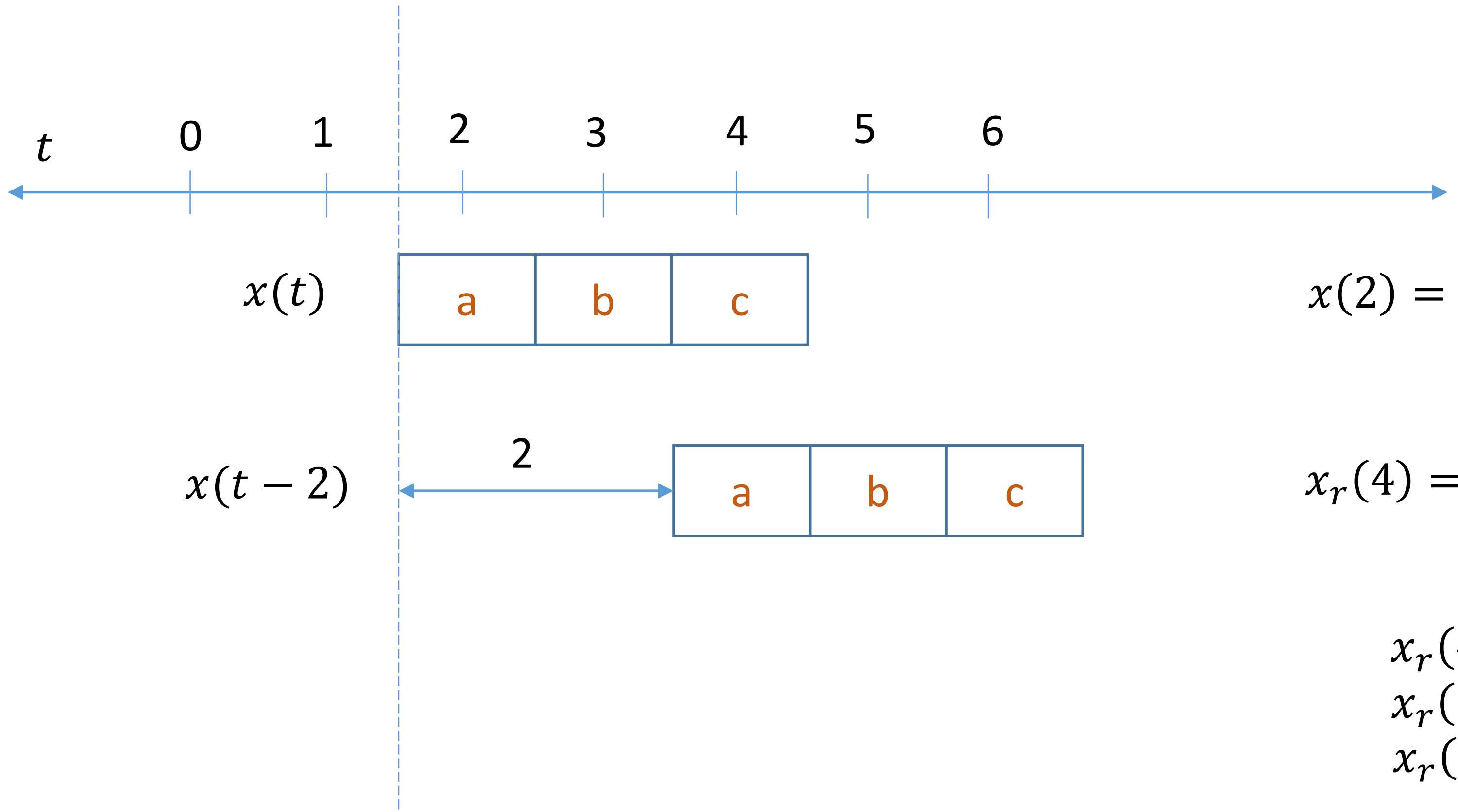


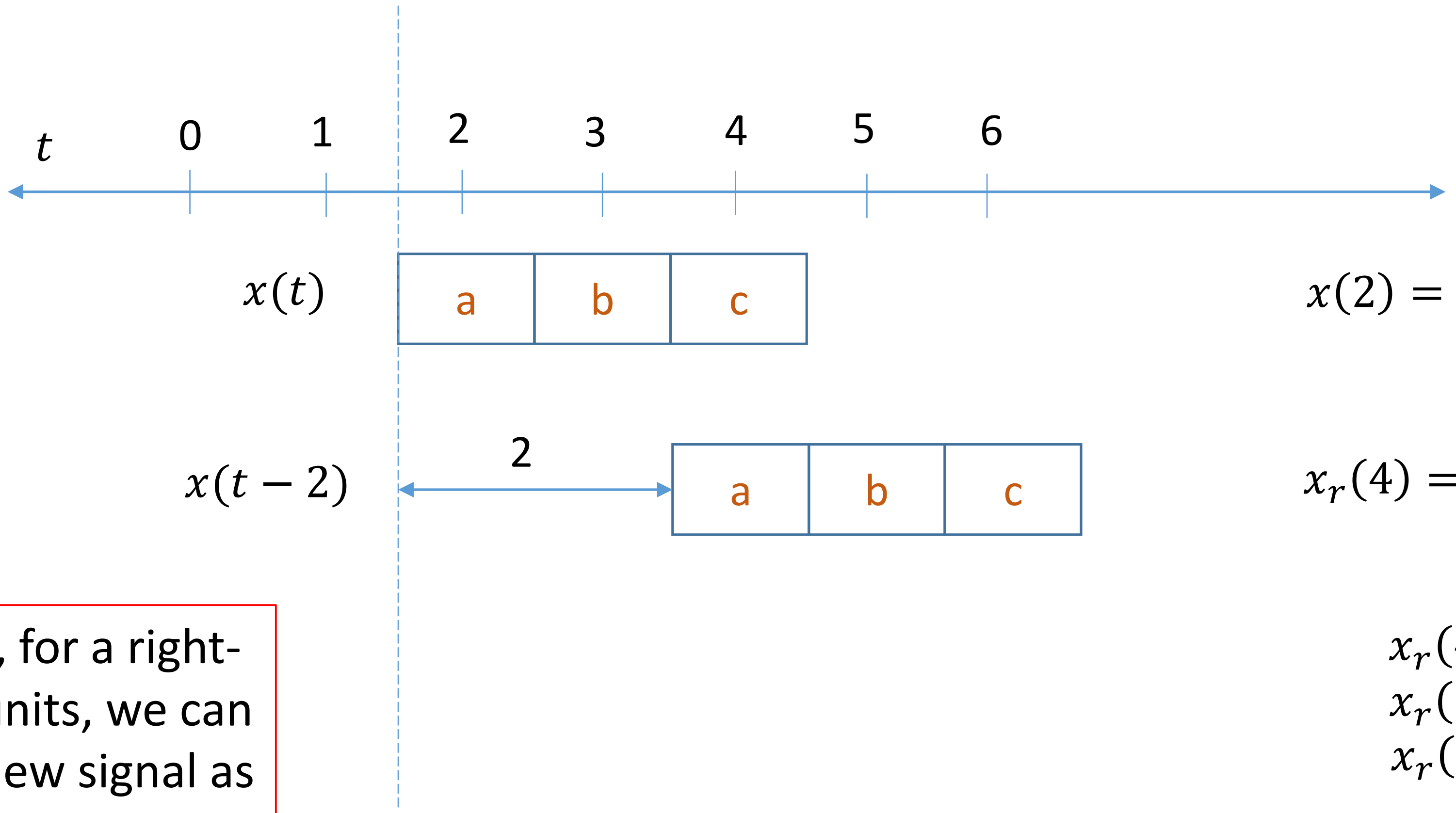
$x(2) = a, x(3) = b, x(4) = c$

$x_r(4) = a, x_r(5) = b, x_r(6) = c$

$$x_r(4) = x(2) = a$$
$$x_r(5) = x(3) = b$$
$$x_r(6) = x(4) = c$$

$$x_r(t) = x(t-2)$$

1. Time-Shifting: How to write mathematically?



$x(2) = a, x(3) = b, x(4) = c$

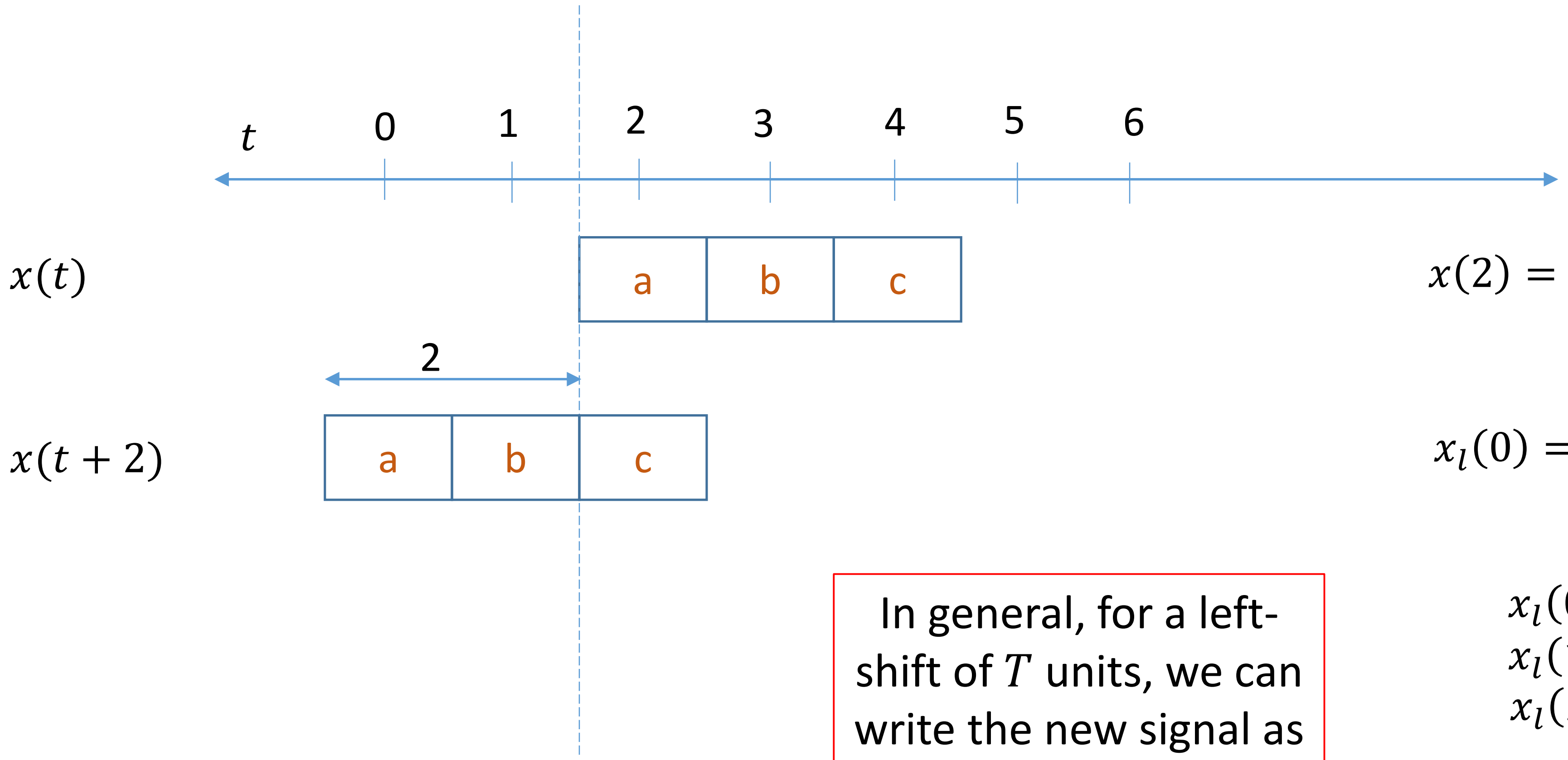
$x_r(4) = a, x_r(5) = b, x_r(6) = c$

In general, for a right-shift of T units, we can write the new signal as
$\boldsymbol{x(t-T)}$

$$x_r(4) = x(2) = a$$
$$x_r(5) = x(3) = b$$
$$x_r(6) = x(4) = c$$

$$x_r(t) = x(t-2)$$

1. Time-Shifting: How to write mathematically?



$x(2) = a, x(3) = b, x(4) = c$

$x_l(0) = a, x_l(1) = b, x_l(2) = c$

In general, for a left-shift of T units, we can write the new signal as
$\boldsymbol{x(t+T)}$

$x_l(0) = x(2) = a$
$x_l(1) = x(3) = b$
$x_l(2) = x(4) = c$

$x_l(t) = x(t+2)$

1. Time-Shifting: How to write mathematically?

$$x(t - T)$$

Assuming $T \geq 0$

Delayed or Advanced? Easy trick to remember:

Put $t = 0$

$$x_s(t) = x(t - T)$$

$$x_s(0) = x(-T)$$

So, whatever happens at 0 now, originally happened at $-T$, which means we are now delayed!!

1. Time-Shifting: How to write mathematically?

$$x(t-T)$$

Assuming $T \geq 0$

Delayed or Advanced? Easy trick to remember:

Put $t = 0$

$$x_s(t) = x(t-T)$$

$$x_s(0) = x(-T)$$

So, whatever happens at 0 now, originally happened at $-T$, which means we are now delayed!!

A funny way to remember: **No body likes delays (i.e., they are a negative thing)**

1. Time-Shifting: How to write mathematically?

$$x(t+T)$$

Assuming $T \geq 0$

Delayed or Advanced? Easy trick to remember:

Put $t = 0$

$$x_{sh}(t) = x(t+T)$$

$$x_{sh}(0) = x(T)$$

So, whatever happens at 0 now, originally happened at T, which means we are now advanced!!

1. Time-Shifting: Example

$$x(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Q. Write the one-second delayed version of this signal

1. Time-Shifting: Example

$$x(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Q. Write the one-second delayed version of this signal

Step 1 $\quad x_d(t) = x(t-1)$

1. Time-Shifting: Example

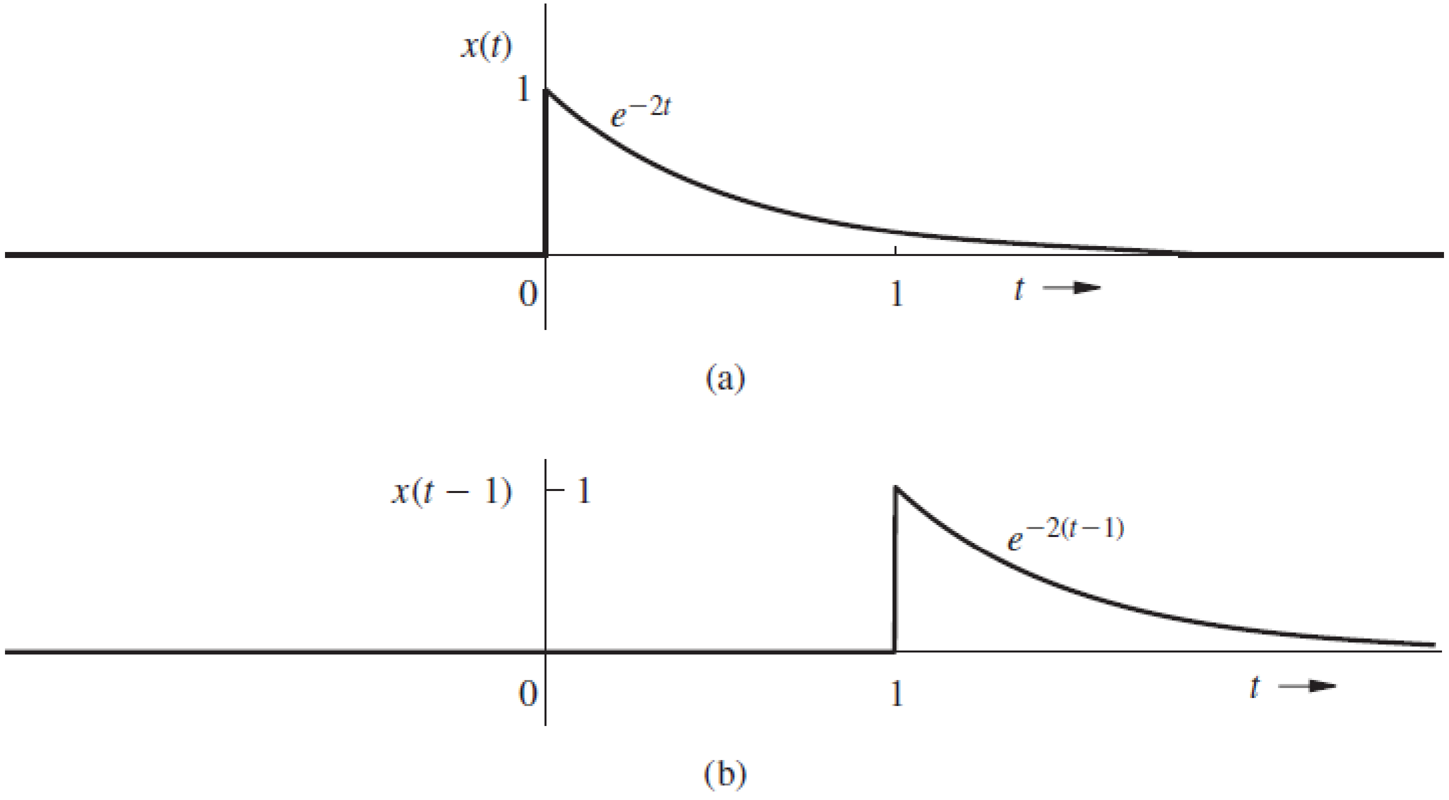
$$x(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Q. Write the one-second delayed version of this signal

Step 1 $$x_d(t) = x(t-1)$$

Step 2 $$x(t-1) = \begin{cases} e^{-2(t-1)} & t-1 \geq 0 \quad \text{or} \quad t \geq 1 \\ 0 & t-1 < 0 \quad \text{or} \quad t < 1 \end{cases}$$

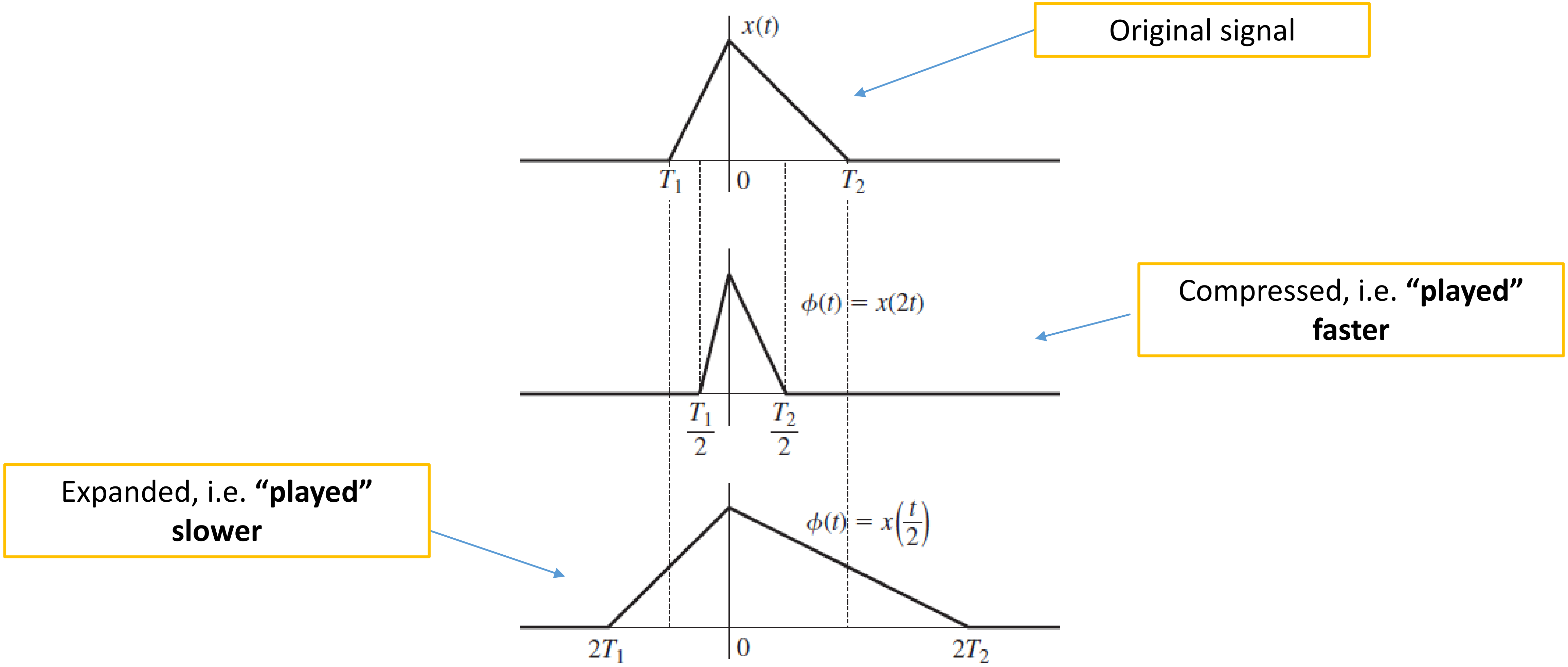
1. Time-Shifting: Example (plotting)



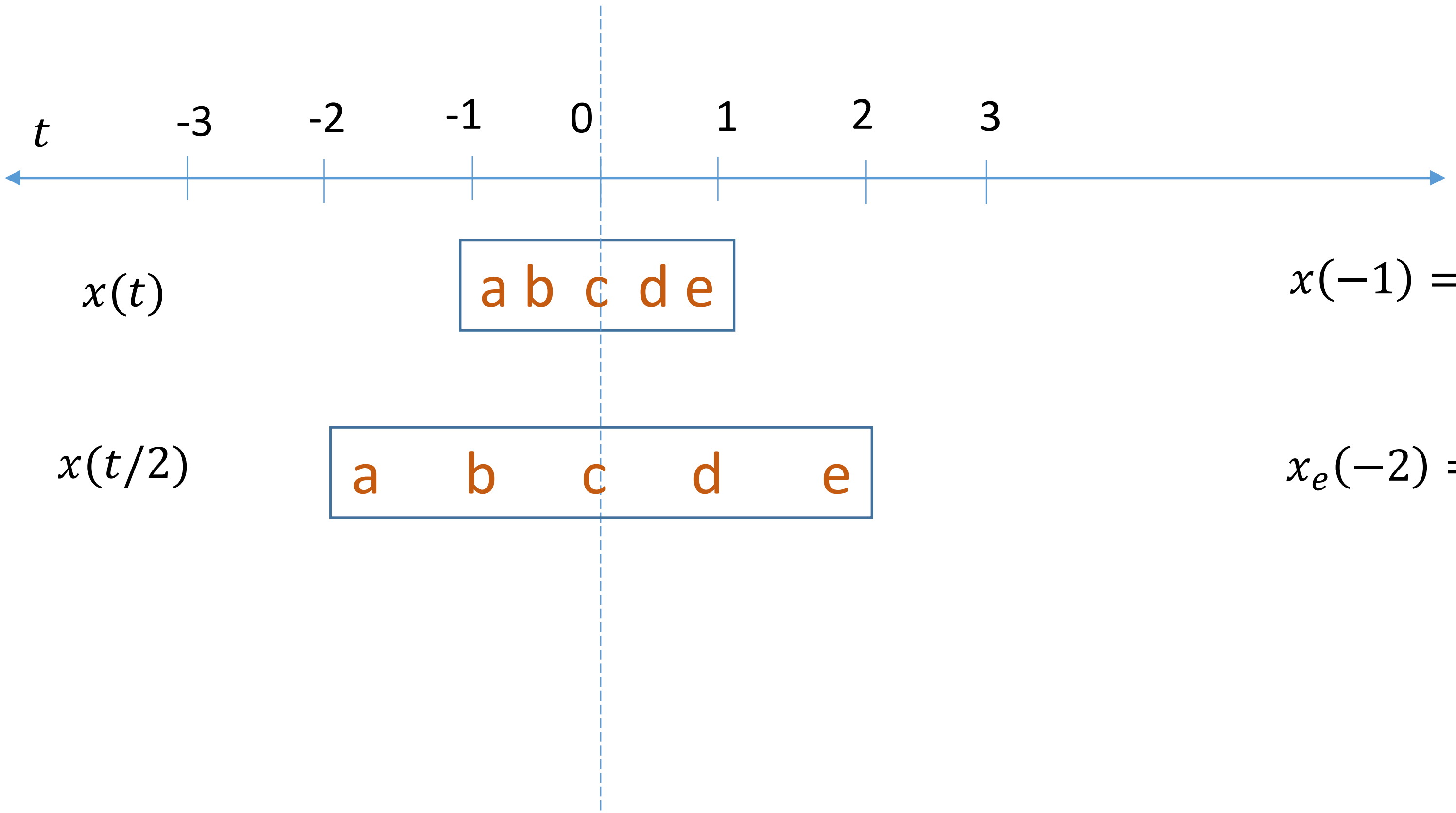
$$x(t) = \begin{cases} e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$x(t-1) = \begin{cases} e^{-2(t-1)} & t \geq 1 \\ 0 & t < 1 \end{cases}$$

2. Time-Scaling



2. Time-Scaling: How to write mathematically?

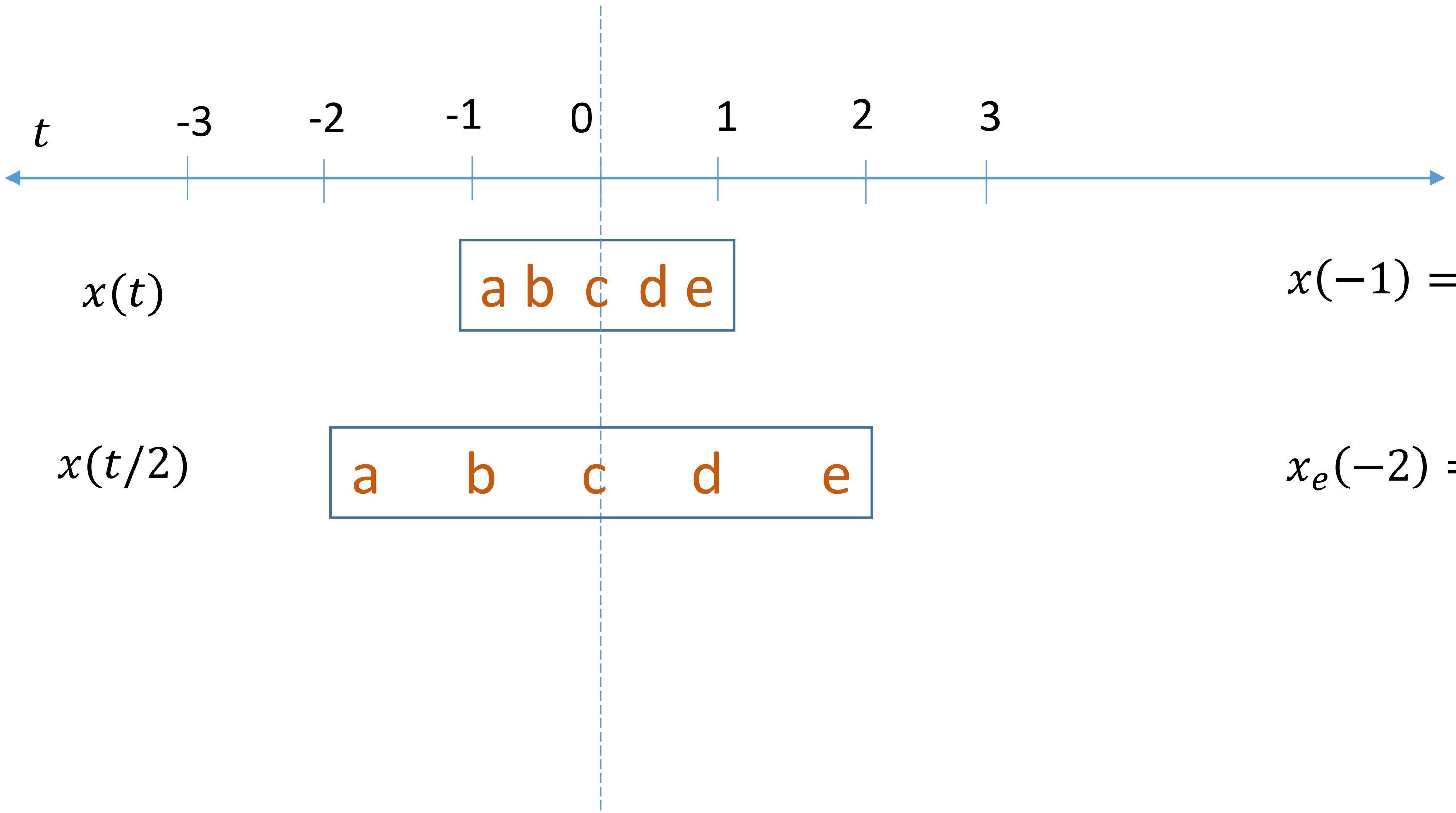


$$x(-1) = a, x(0) = c, x(1) = e$$

$$x_e(-2) = a, x_e(0) = c, x_e(2) = e$$

$$x_e(-2) = x(-1) = a$$
$$x_e(0) = x(0) = c$$
$$x_e(2) = x(1) = e$$

2. Time-Scaling: How to write mathematically?



$x(-1) = a, x(0) = c, x(1) = d$

$x_e(-2) = a, x_e(0) = c, x_e(2) = e$

$$x_e(-2) = x(-1) = a$$
$$x_e(0) = x(0) = c$$
$$x_e(2) = x(1) = e$$

$$x_e(t) = x(t/2)$$

2. Time-Scaling: How to write mathematically?

In general, for a time-scaling (**expansion**) by a factor $a > 1$, we can write the expanded signal as

$$x(t/a)$$

And for a time-scaling (**compression**) by a factor $a > 1$, we can write the compressed signal as

$$x(at)$$

2. Time-Scaling: Example

$$x(t) = \begin{cases} 2 & -1.5 \le t < 0 \\ 2e^{-t/2} & 0 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

Q. Write a compressed version of the signal, with compression factor $a = 3$.

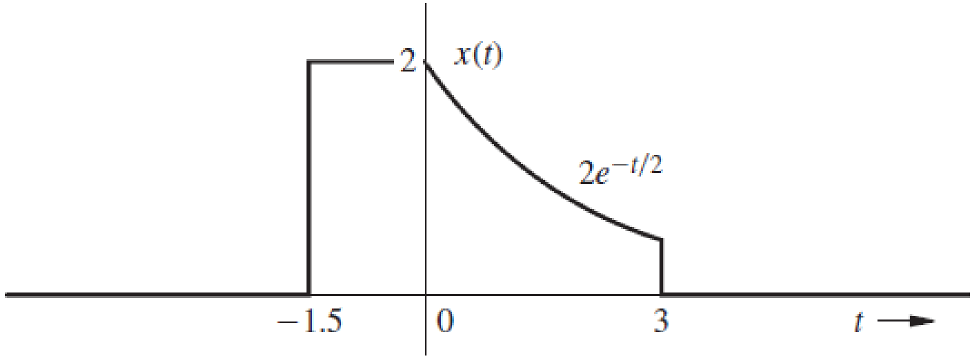
Step 1

$$x_c(t) = x(3t)$$

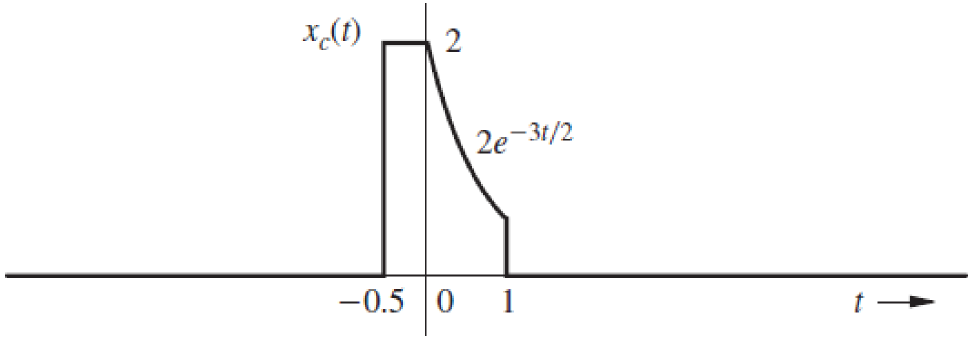
Step 2

$$x(3t) = \begin{cases} 2 & -1.5 \le 3t < 0 \quad \text{or} \quad -0.5 \le t < 0 \\ 2e^{-3t/2} & 0 \le 3t < 3 \quad \text{or} \quad 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Time-Scaling: Example (plotting)



(a)



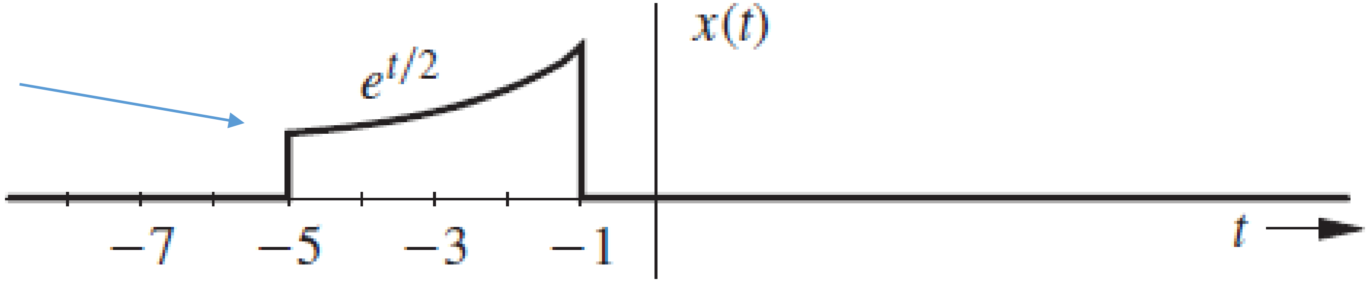
(b)

$$x(t) = \begin{cases} 2 & -1.5 \le t < 0 \\ 2\,e^{-t/2} & 0 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$x(3t) = \begin{cases} 2 & -1.5 \le 3t < 0 \quad \text{or} \quad -0.5 \le t < 0 \\ 2\,e^{-3t/2} & 0 \le 3t < 3 \quad \text{or} \quad 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

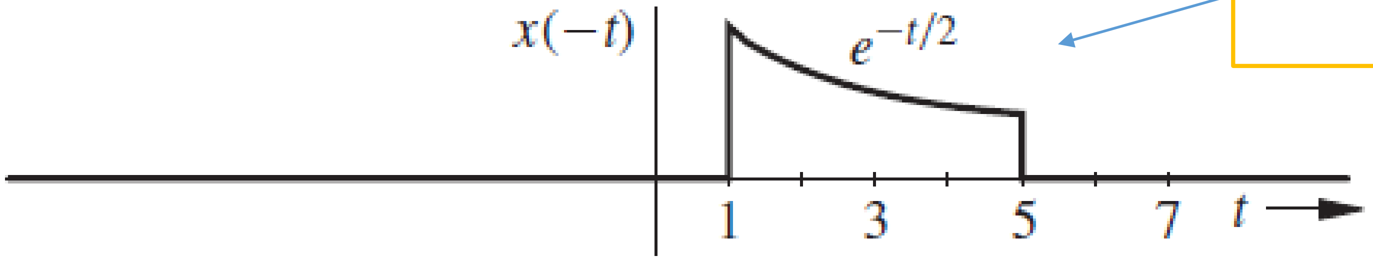
3. Time-Reversal

Original signal



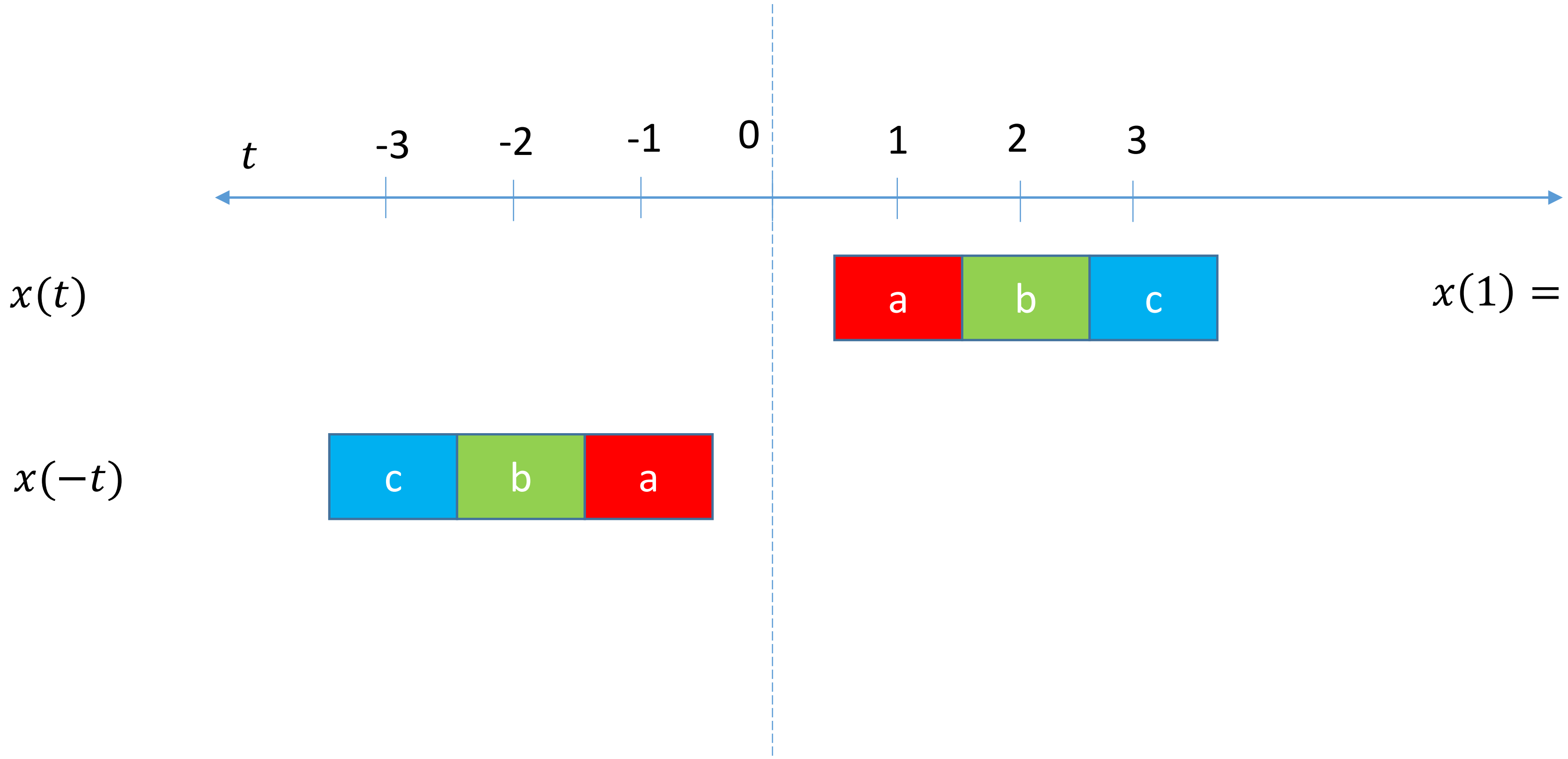
(a)

Time-reversed, i.e. **“flipped” horizontally**



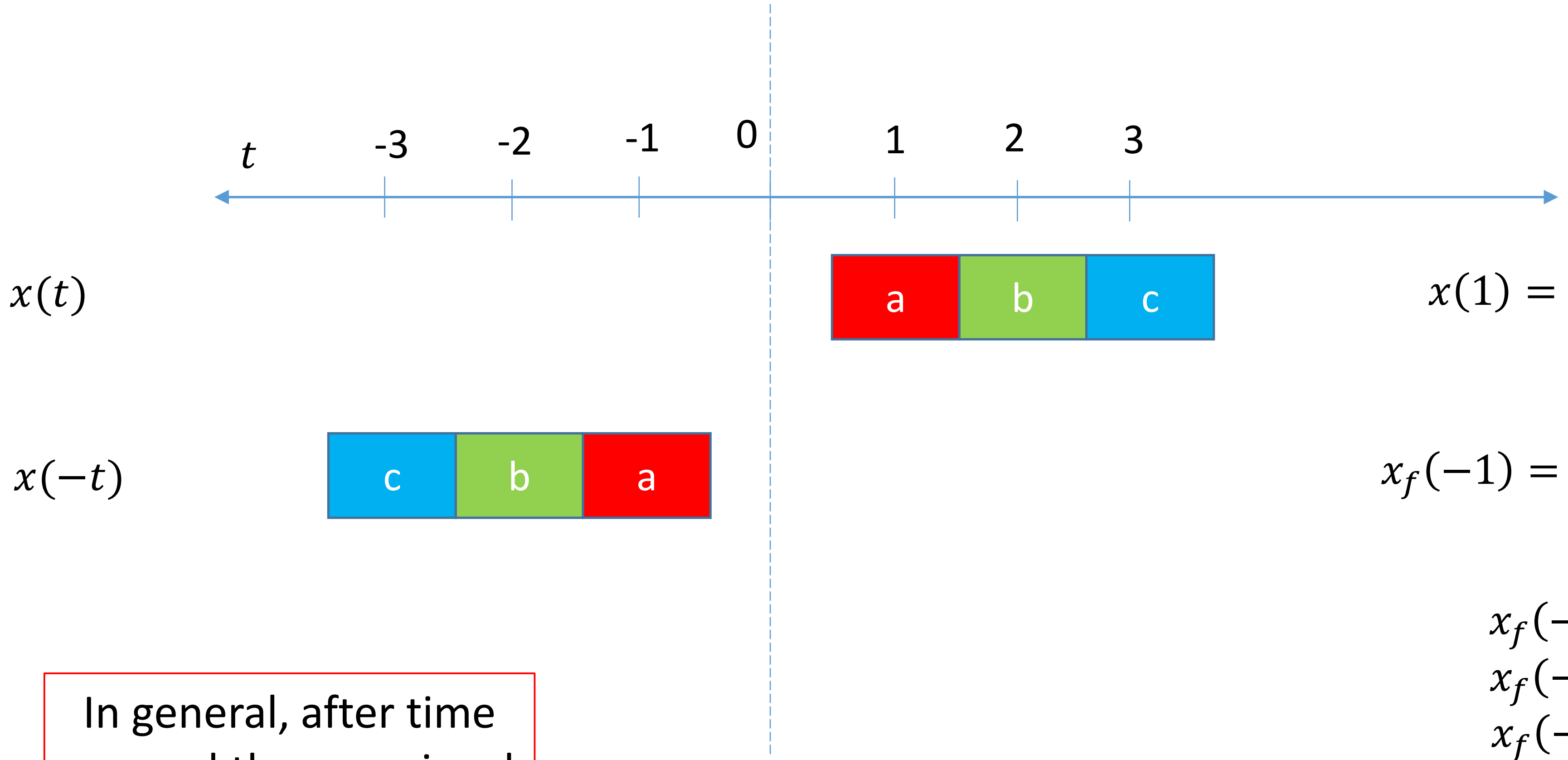
(b)

3. Time-Reversal: How to write mathematically?



$x(1) = a, x(2) = b, x(3) = c$

3. Time-Reversal: How to write mathematically?



$$x(1) = a, x(2) = b, x(3) = c$$

$$x_f(-1) = a, x_f(-2) = b, x_f(-3) = c$$

$$x_f(-1) = x(1) = a$$
$$x_f(-2) = x(2) = b$$
$$x_f(-3) = x(3) = c$$

$$x_f(t) = x(-t)$$

In general, after time reversal the new signal can be written as $\boldsymbol{x(-t)}$

3. Time-Reversal: Example

$$x(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases}$$

Q. Write the time-reversed version of the given signal.

3. Time-Reversal: Example

$$x(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases}$$

Q. Write the time-reversed version of the given signal.

Step 1 $\quad x_f(t) = x(-t)$

3. Time-Reversal: Example

$$x(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases}$$

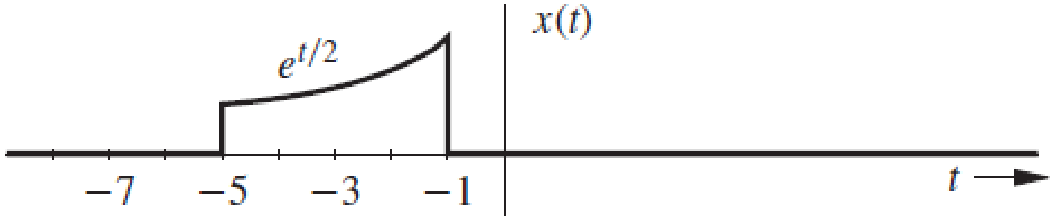
Q. Write the time-reversed version of the given signal.

Step 1 $\quad x_f(t) = x(-t)$

Step 2

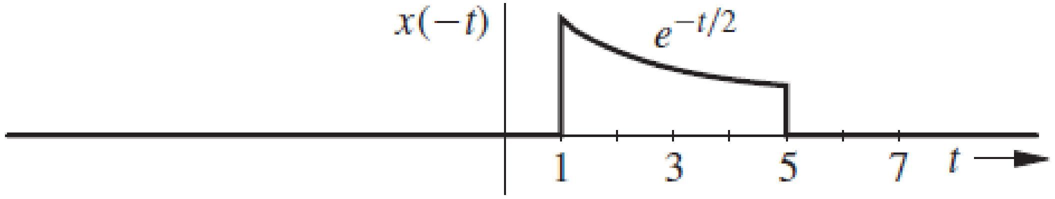
$$x(-t) = \begin{cases} e^{-t/2} & -1 \geq -t > -5 \quad \text{or} \quad 1 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}$$

3. Time-Reversal: Example (plotting)



(a)

$$x(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases}$$

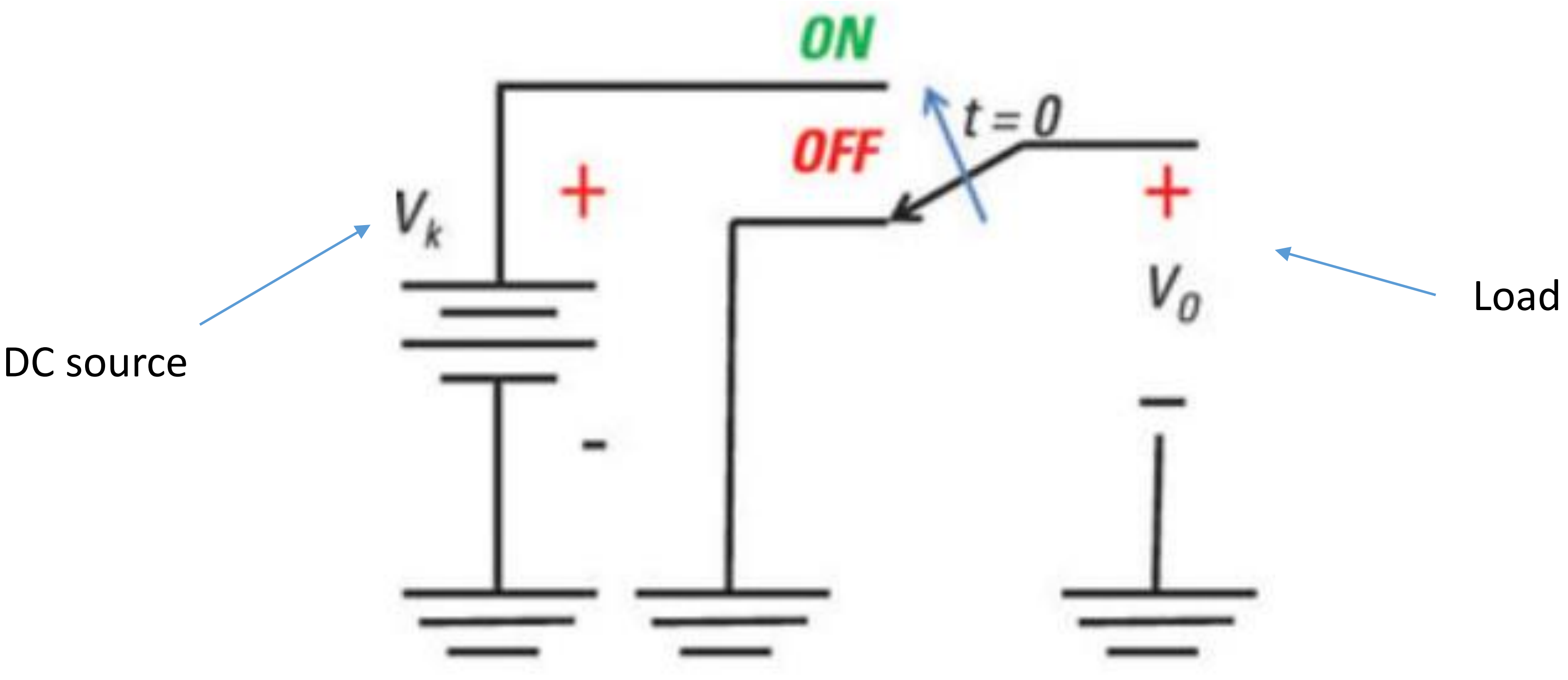


(b)

$$x(-t) = \begin{cases} e^{-t/2} & -1 \geq -t > -5 \quad \text{or} \quad 1 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}$$

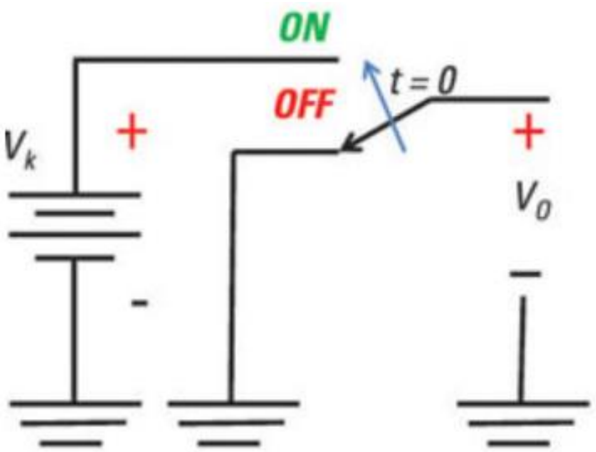
It is good to be intimately familiar with some signals that show up again and again and again and...

1.1 Unit Step

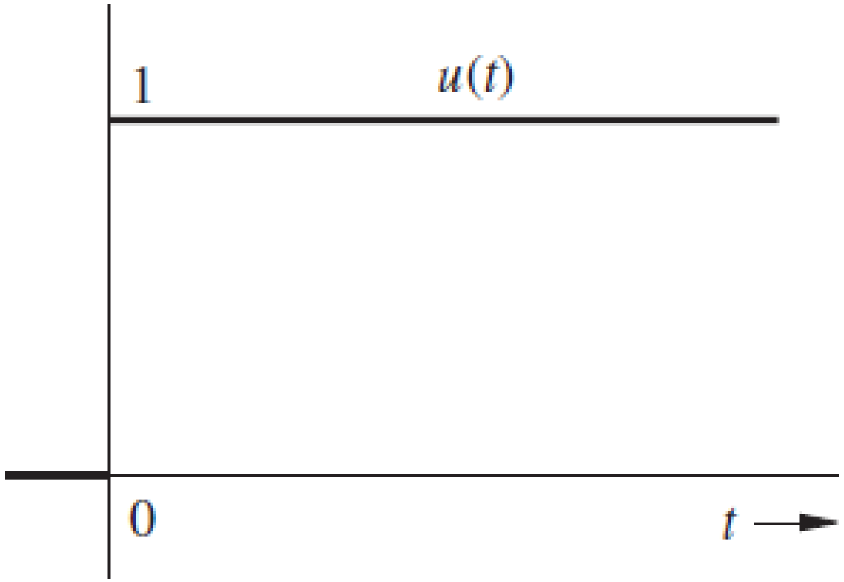


Circuit approximation of step

1.1 Unit Step: Graph & Equation



Circuit approximation of step

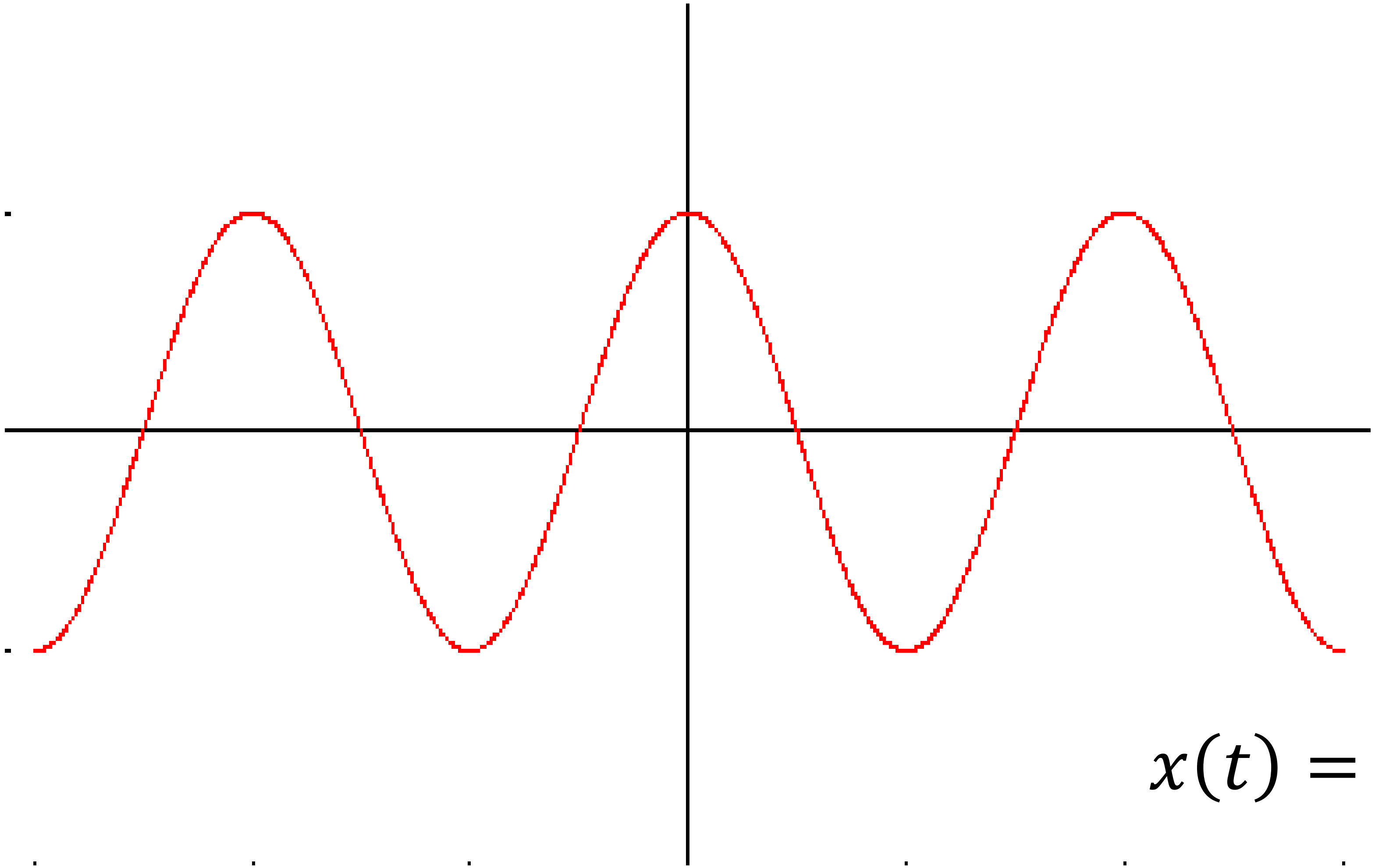


$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Q. How can we limit a signal so it doesn't start before $t = 0$?

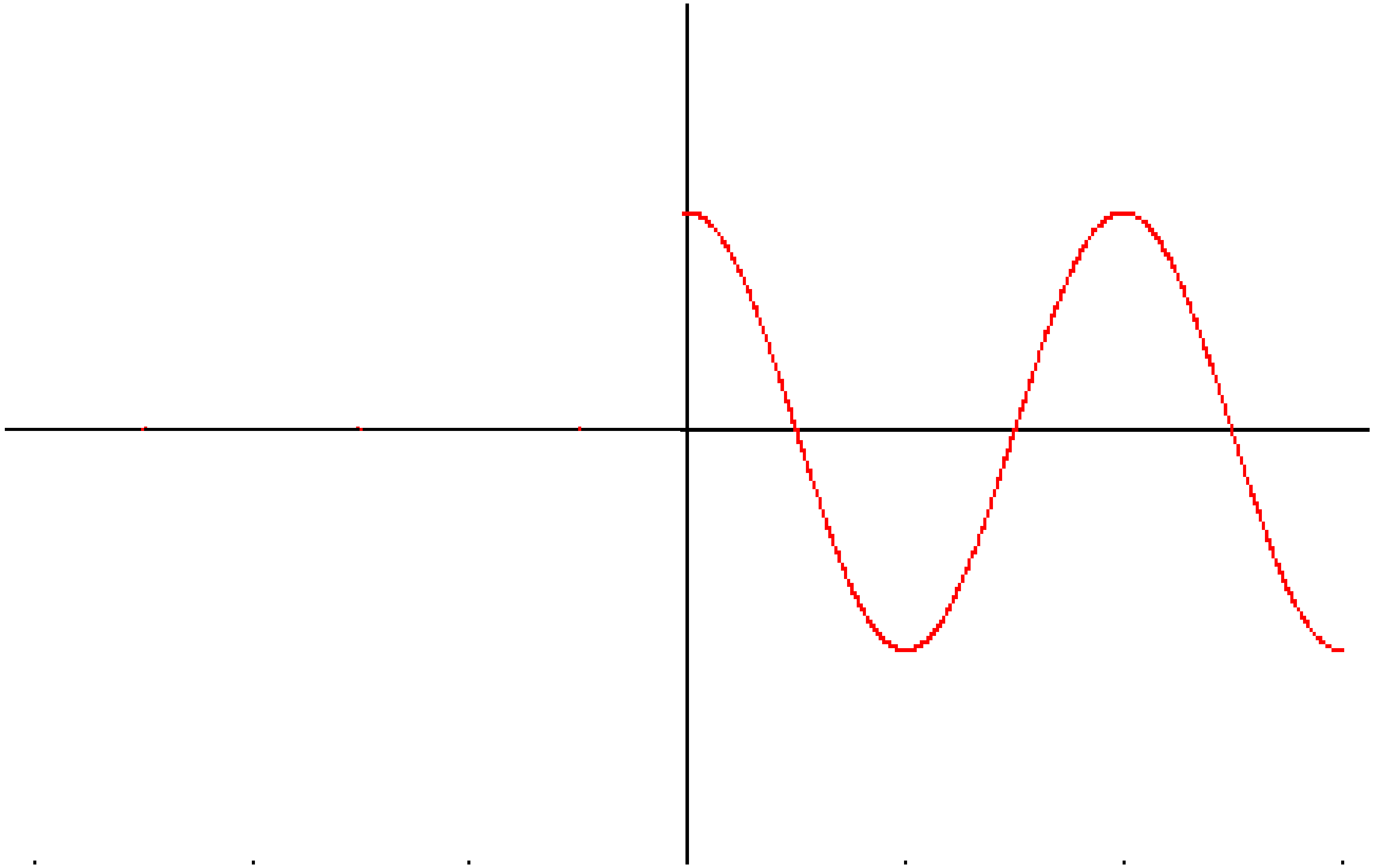
Answer: multiply it with unit step!!

1.2. Multiplication with a Unit Step

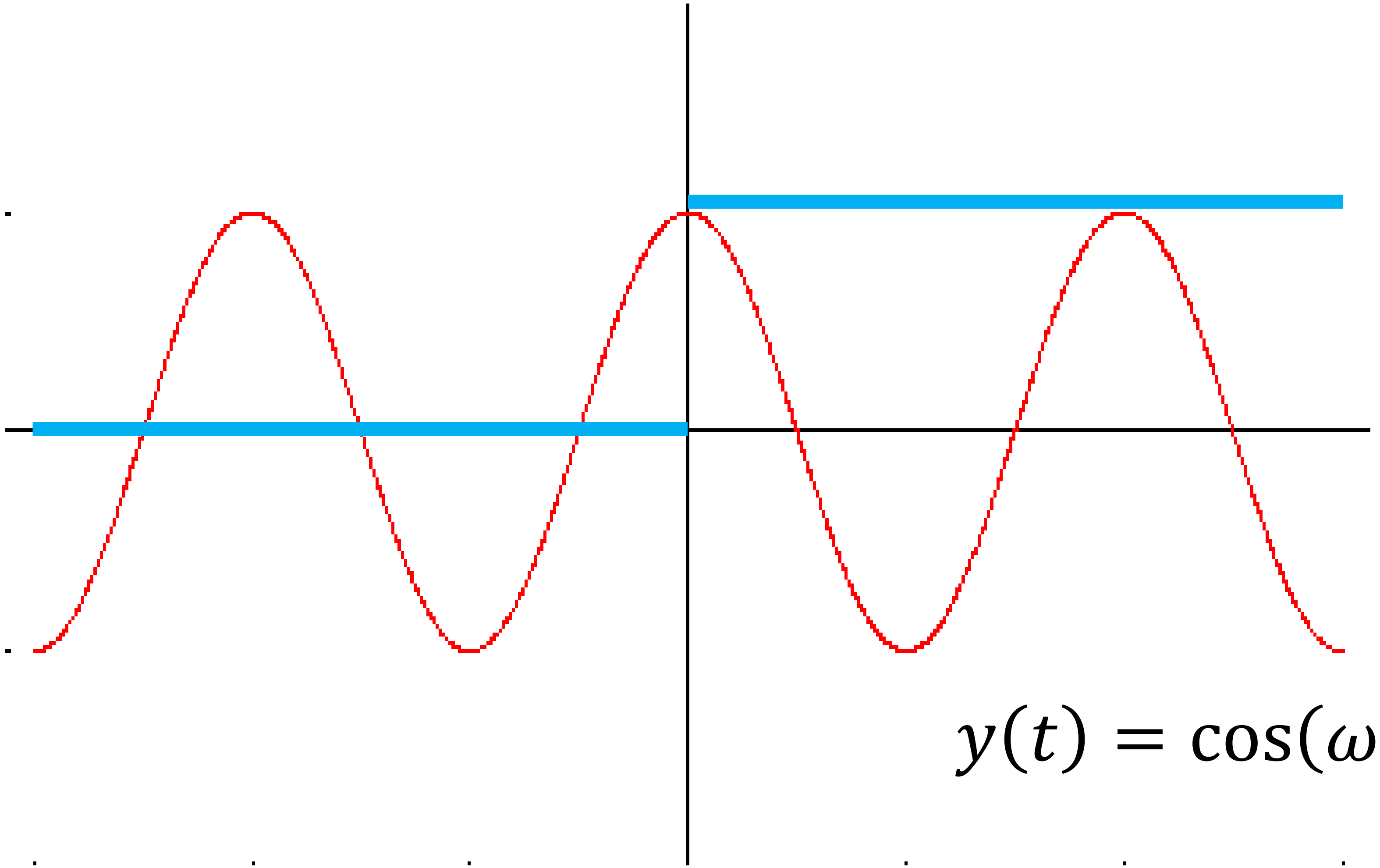


$$x(t) = \cos(\omega t)$$

1.2 Multiplication with a Unit Step

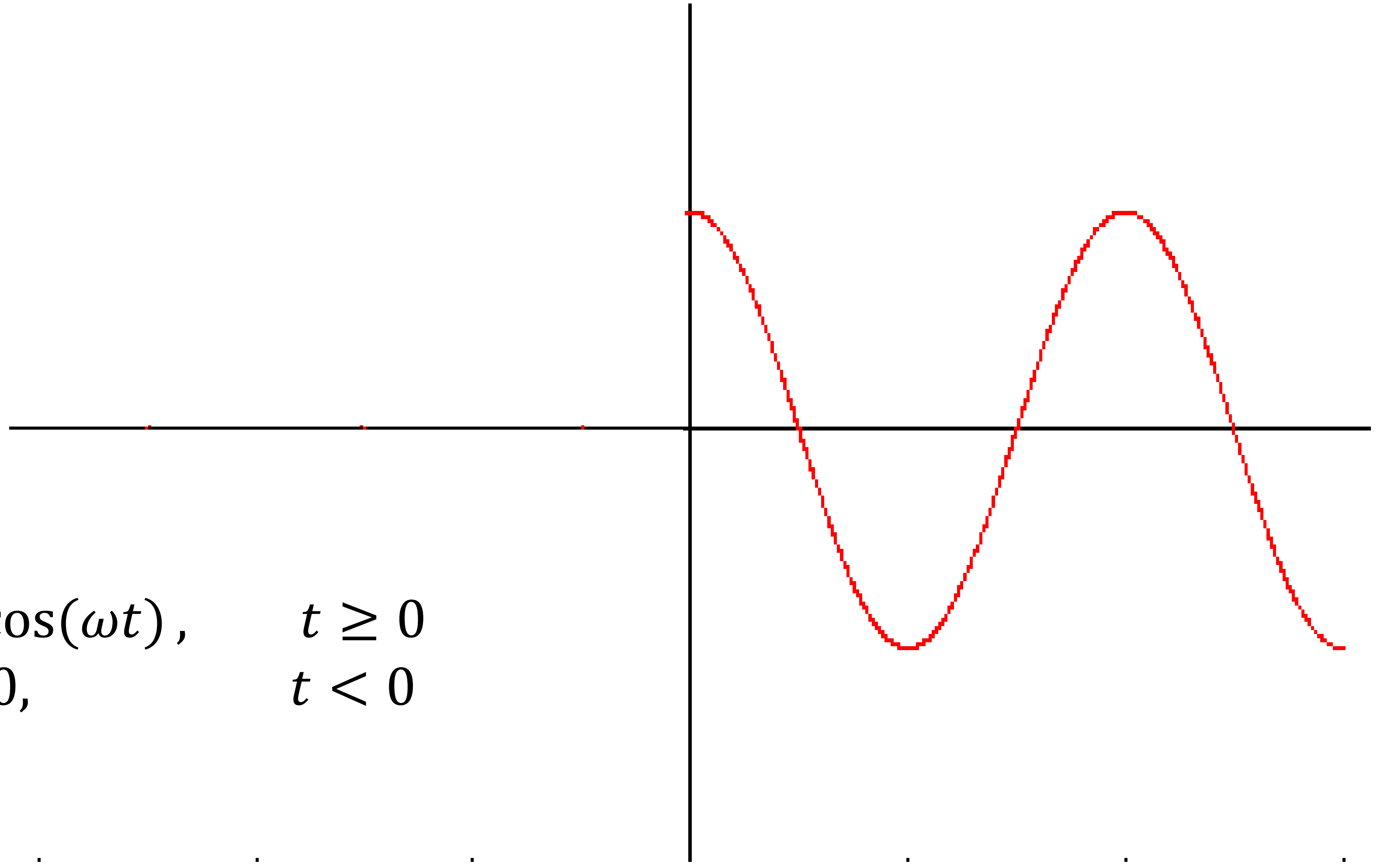


1.2 Multiplication with a Unit Step



$$y(t) = \cos(\omega t) \times u(t)$$

1.2 Multiplication with a Unit Step



$$y(t) = \begin{cases} \cos(\omega t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

1.3 Writing a Piece-Wise Function in terms of Unit Step

$$x(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$x(t) = e^{-at}u(t)$$

1.4 Time-Shifting a Unit Step

$$u(t-T)=\begin{cases}1, & t-T\geq 0\\ 0, & t-T<0\end{cases}$$

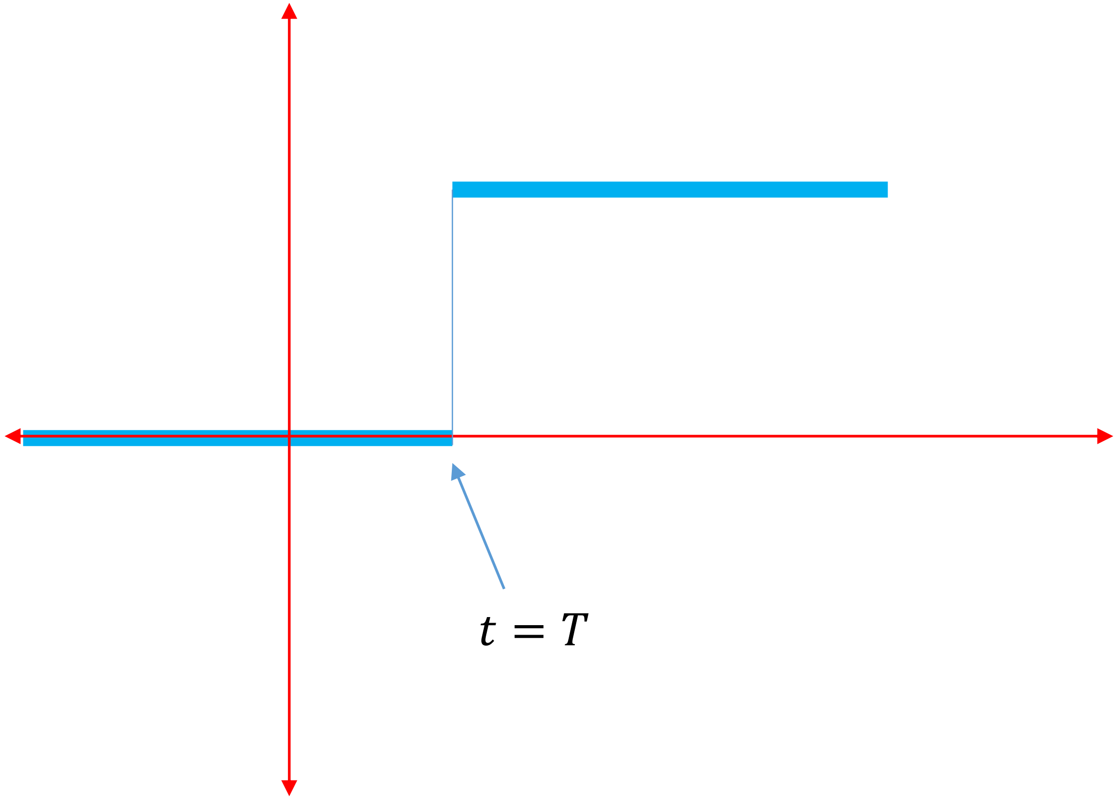
$$\Big\Downarrow$$

$$u(t-T)=\begin{cases}1, & t\geq T\\ 0, & t<T\end{cases}$$

1.4 Time-Shifting a Unit Step

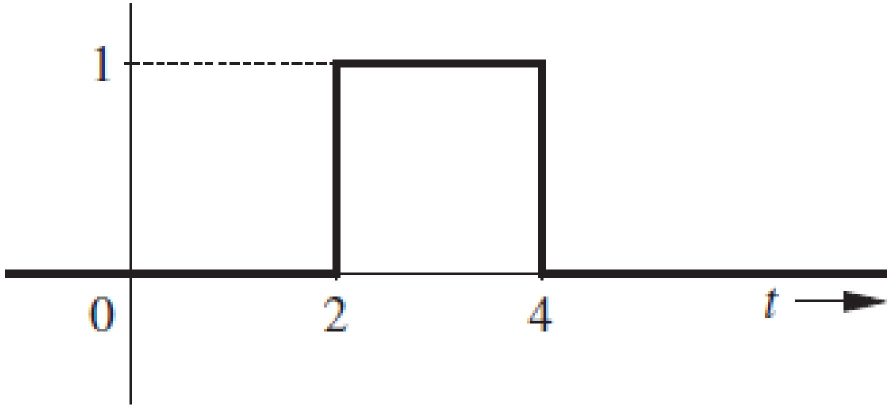
$$u(t-T) = \begin{cases} 1, & t-T \geq 0 \\ 0, & t-T < 0 \end{cases}$$

$$u(t-T) = \begin{cases} 1, & t \geq T \\ 0, & t < T \end{cases}$$



1.5 Making off-on-off (rectangular pulse) with Unit Step

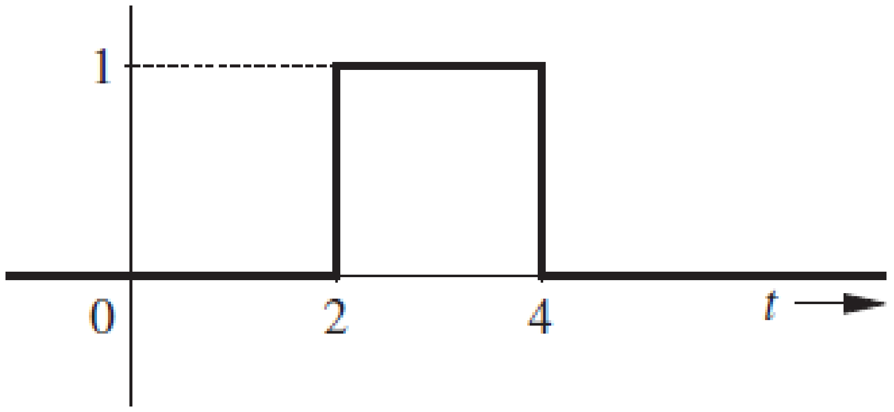
$$x(t) = u(t-2) - u(t-4)$$



$$u(t-2) = \begin{cases} 1, & t \geq 2 \\ 0, & t < 2 \end{cases}$$

1.5 Making off-on-off (rectangular pulse) with Unit Step

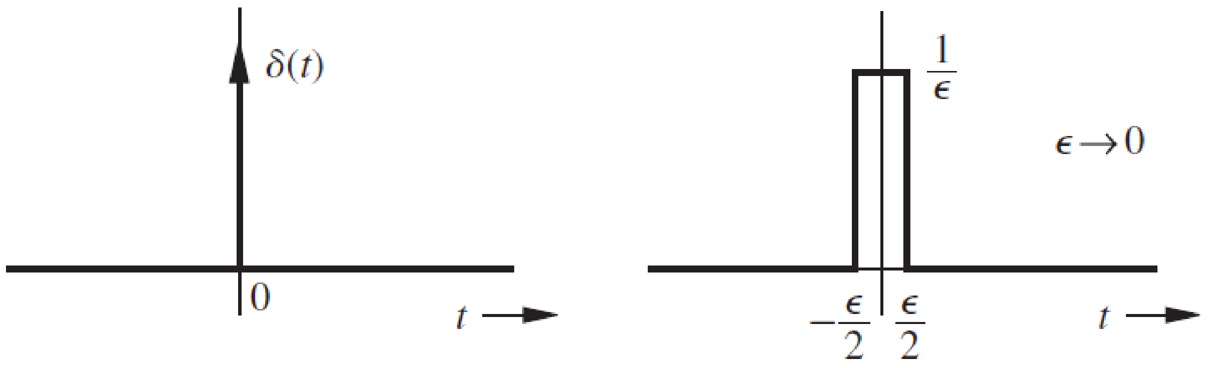
$$u(t-2) = \begin{cases} 1, & t \geq 2 \\ 0, & t < 2 \end{cases}$$



$$-u(t-4) = \begin{cases} -1, & t \geq 4 \\ 0, & t < 4 \end{cases}$$

$$x(t) = u(t-2) - u(t-4) = \begin{cases} 0, & t < 2 \\ 1+0, & 2 \leq t < 4 \\ 1-1, & t \geq 4 \end{cases}$$

2.1 Unit Impulse Function

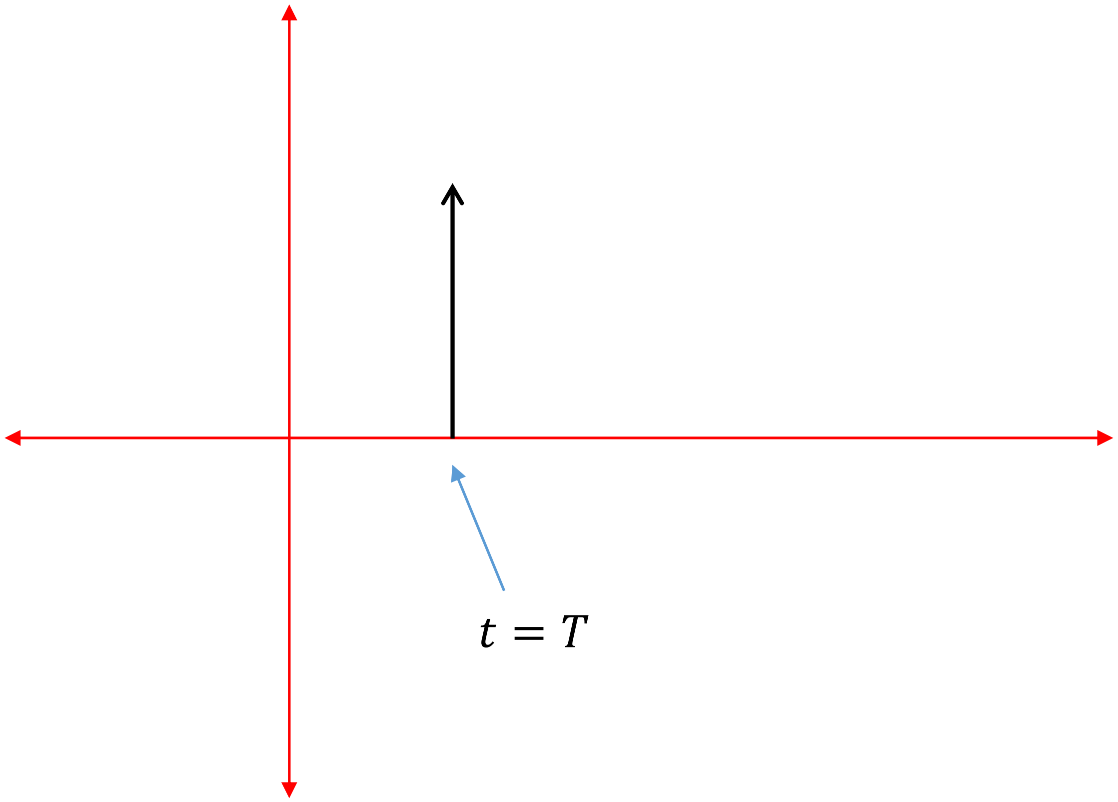


$$\delta(t) = 0 \quad t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

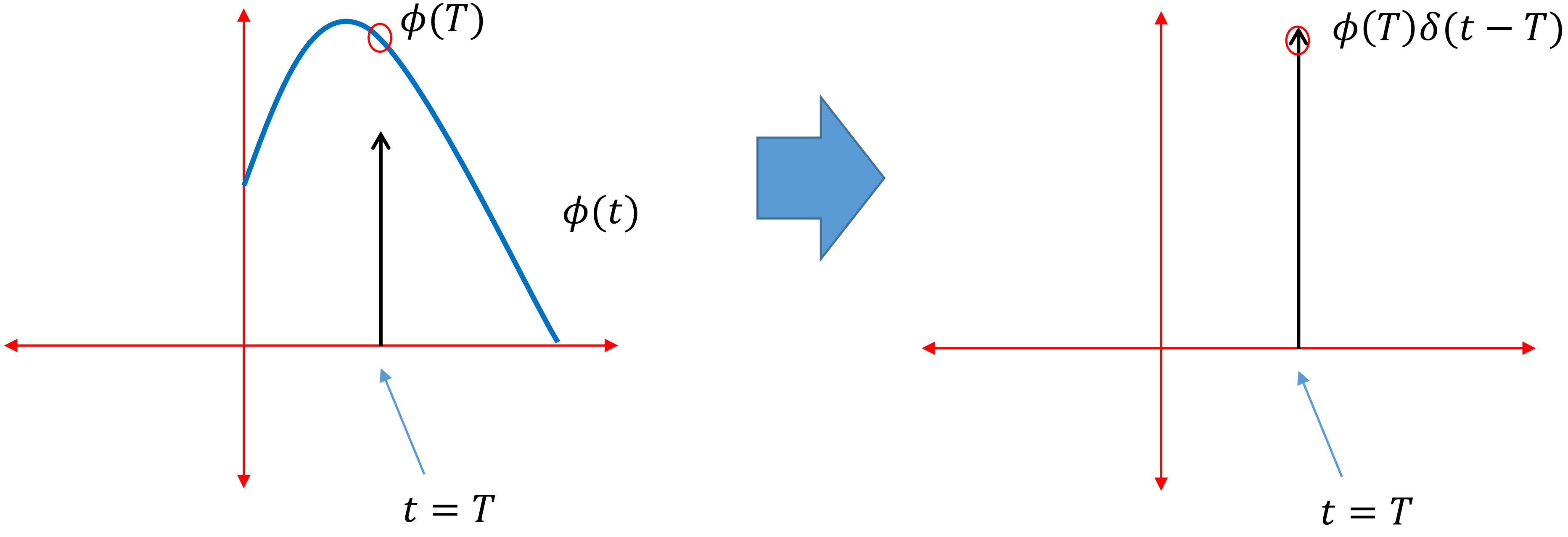
2.1 Time-Shifting a Unit Impulse

$$\delta(t - T) = 0 \quad t - T \neq 0$$

$$\delta(t - T) = 0 \quad t \neq T$$



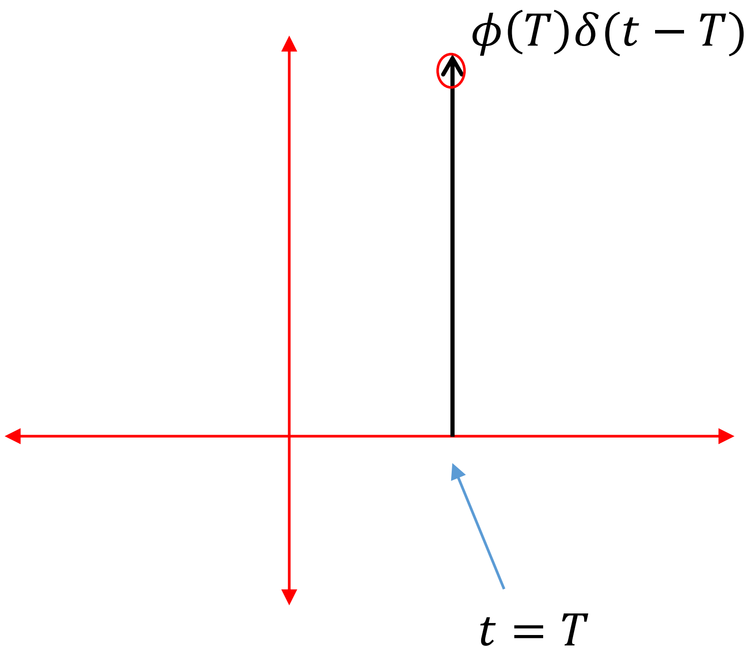
2.2 Multiplying a Function with Unit Impulse

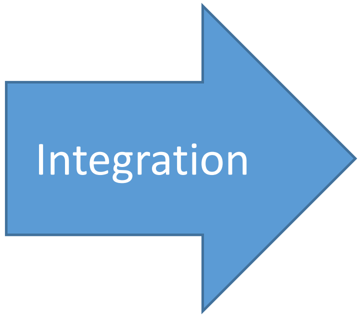


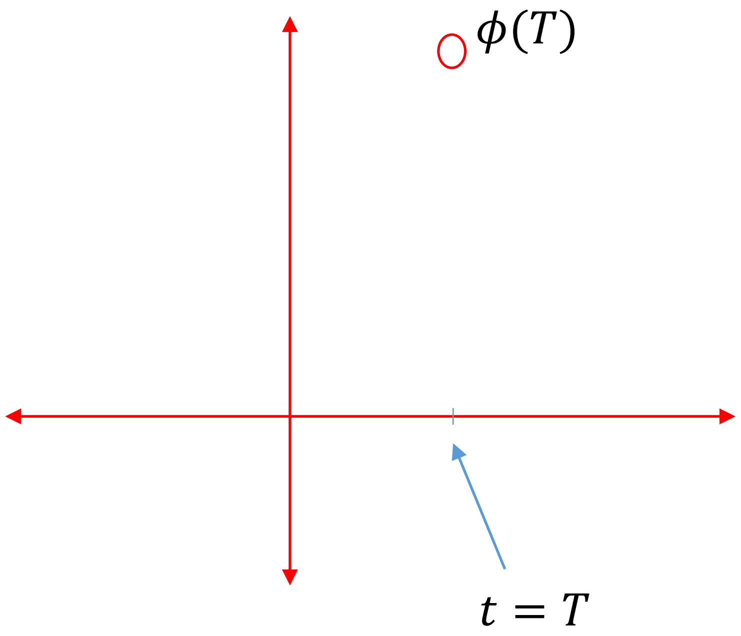
$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

Impulse scaled to $\phi(T)$

2.3 Sampling Property of Unit Impulse







$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)\,dt = \phi(T)\int_{-\infty}^{\infty} \delta(t)\,dt = \phi(T)$$

So we get the value/sample of the function $\phi(t)$ at T.

2.4 Relation Between Unit Step & Unit Impulse

$$\frac{du(t)}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

2.5 Examples

Show that $(t^3 + 3)\delta(t) = 3\delta(t)$

Let $\phi(t) = t^3 + 3$, then using: $\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$

For us $T = 0$, which gives $\phi(t)\delta(t) = \phi(0)\delta(t)$

$$(t^3 + 3)\delta(t) = (0^3 + 3)\delta(t) = 3\delta(t)$$

2.6 Examples

Show that

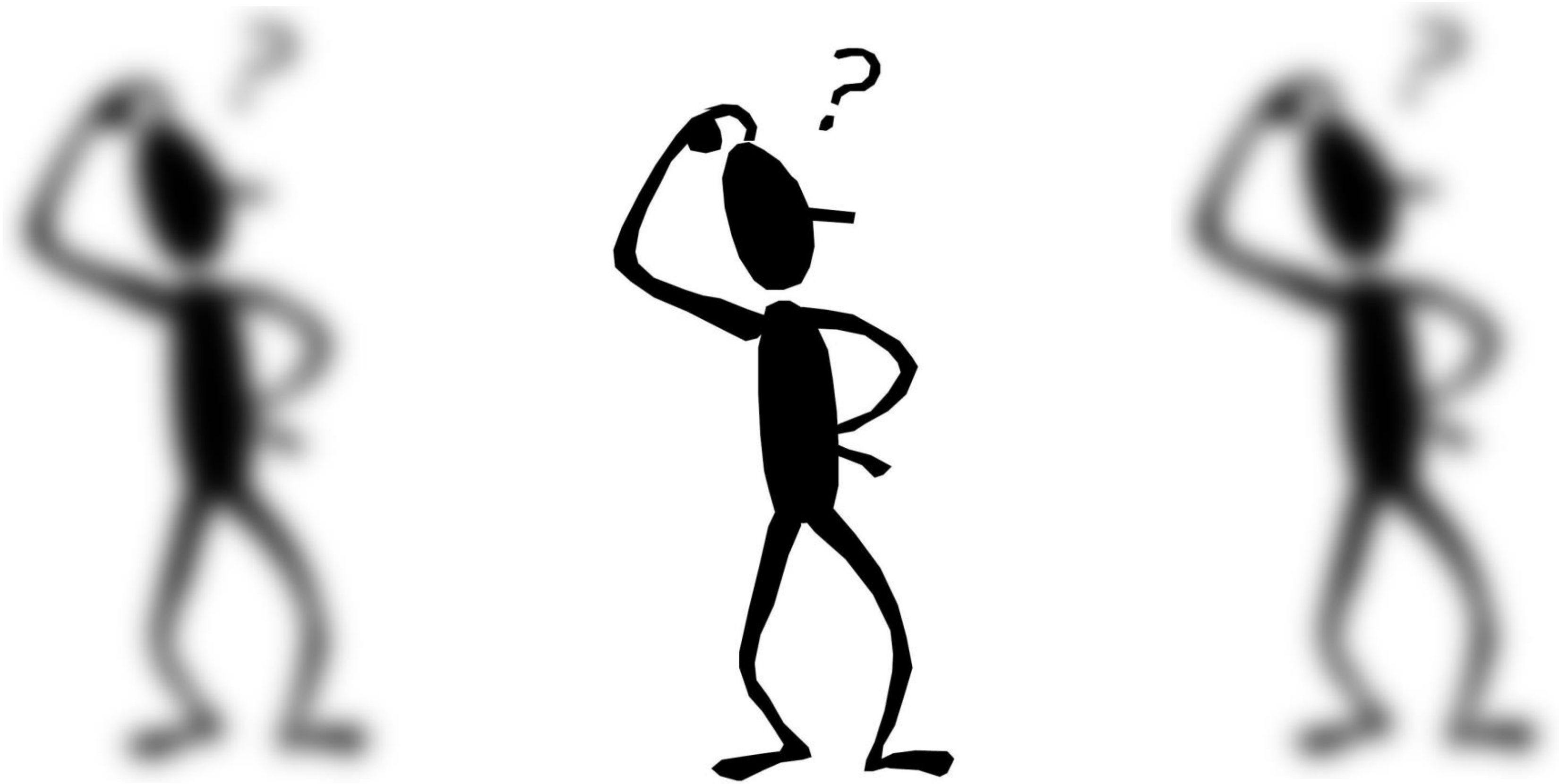
(a) $\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt = 1$

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)\,dt = \phi(T)\int_{-\infty}^{\infty} \delta(t)\,dt = \phi(T)$$

Use this, with $\phi(t) = e^{-j\omega t}$ and $T = 0$

$$\phi(0) = e^{-j\omega \times 0} = 1$$

Questions?? Thoughts??



ES 332
Signals and Systems

with

Dr. Naveed R. Butt

@

GIKI - FES

Signal Basics III

Complex Numbers

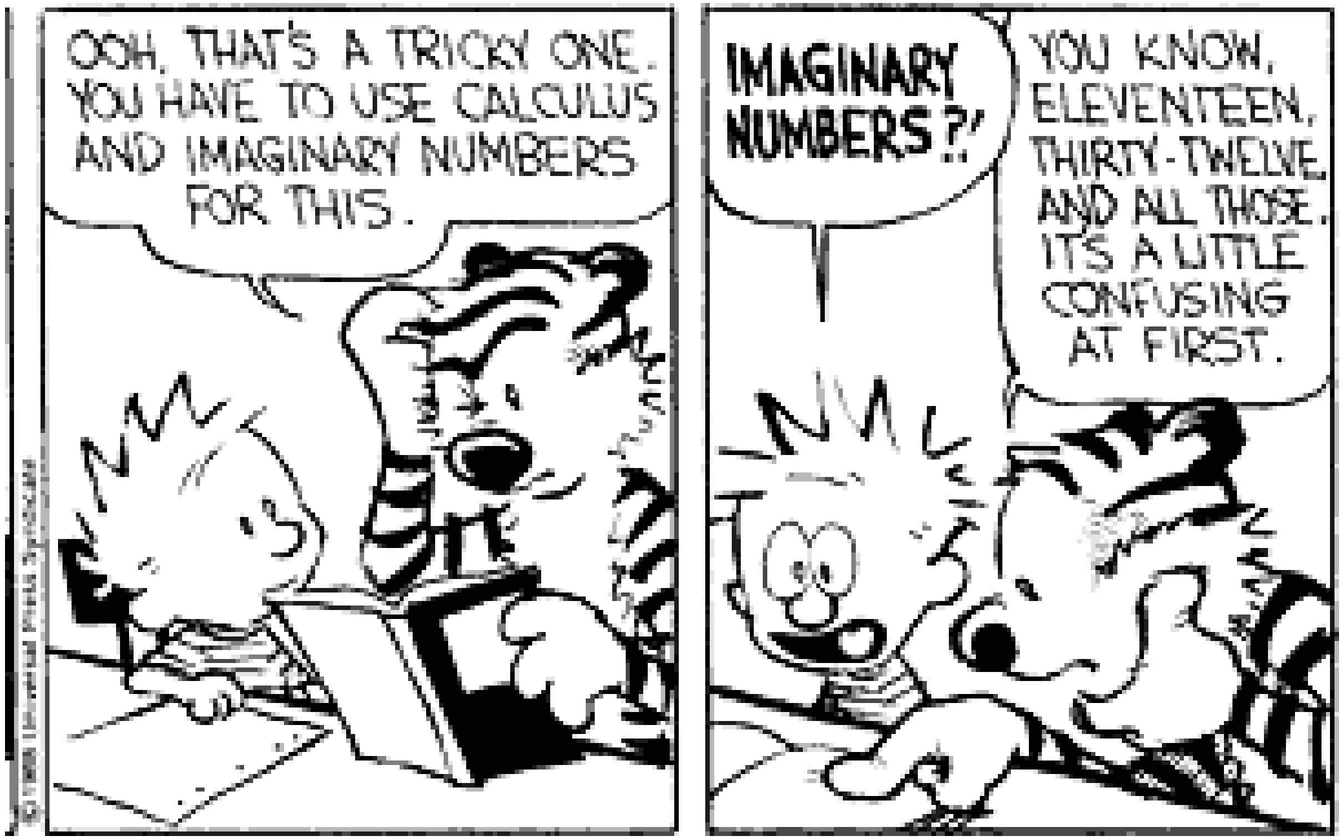
Quick Revision

Models

Complex Exponential

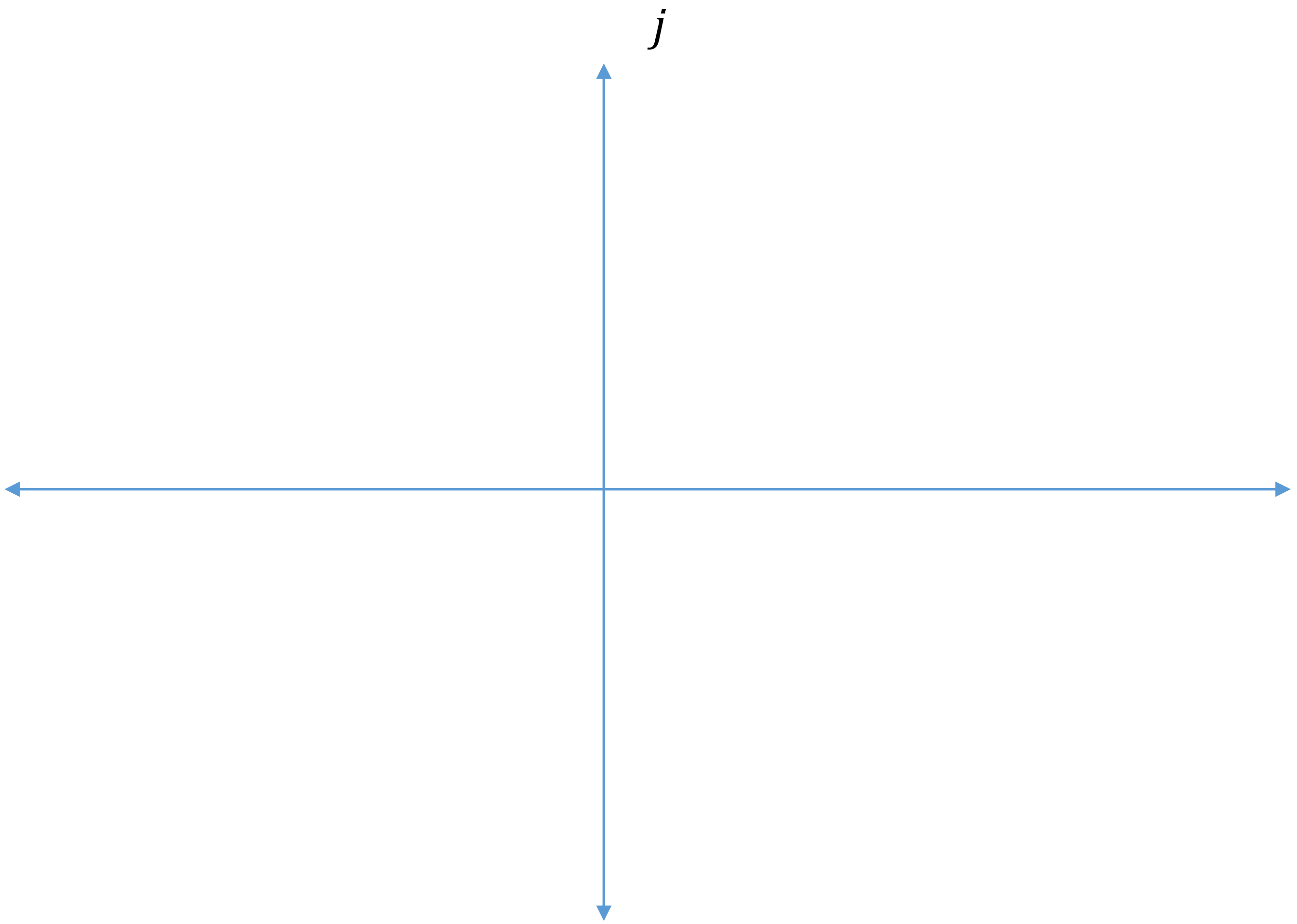
Examples

Some practice problems

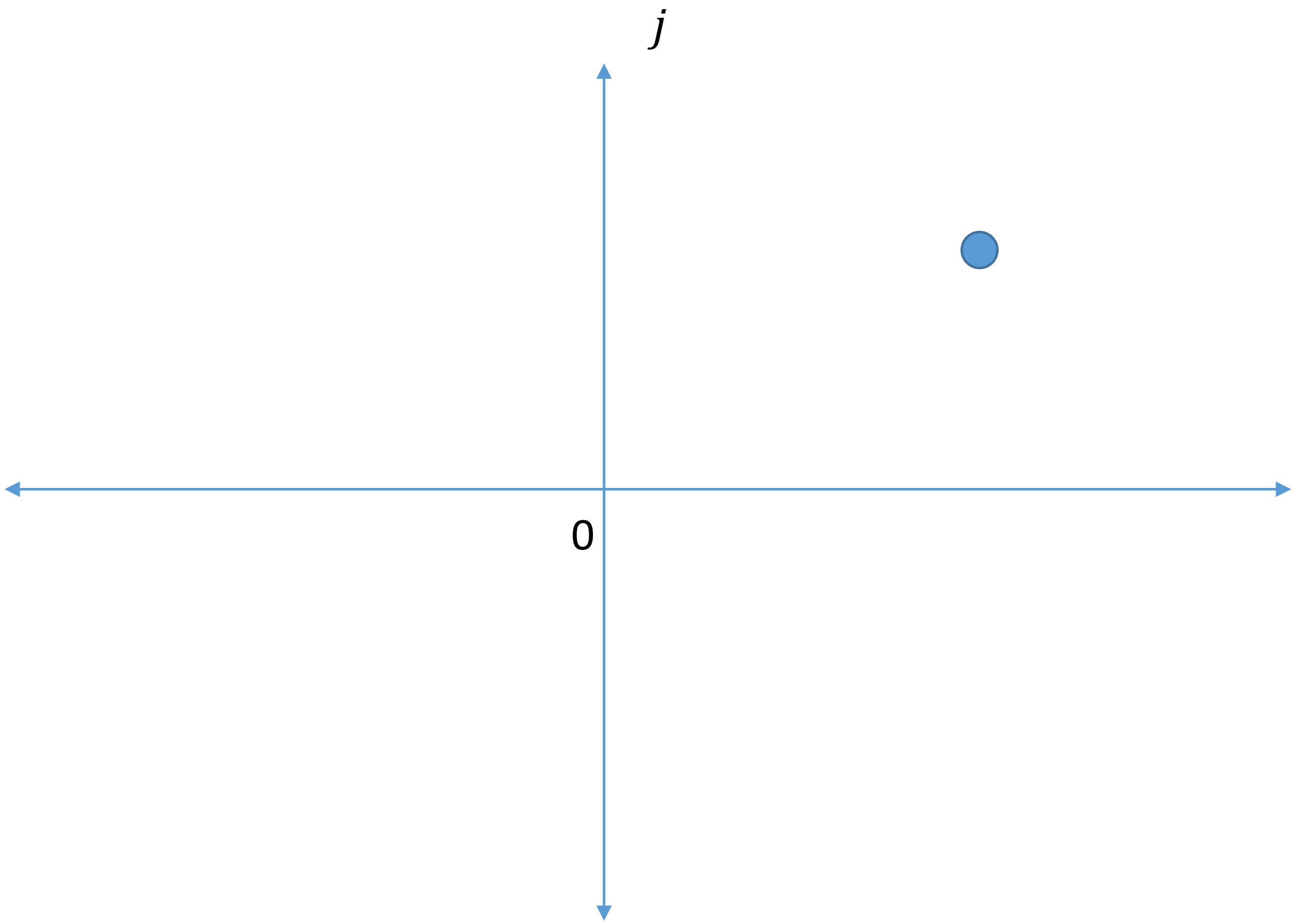

OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE, AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.

Plotting a Complex Number

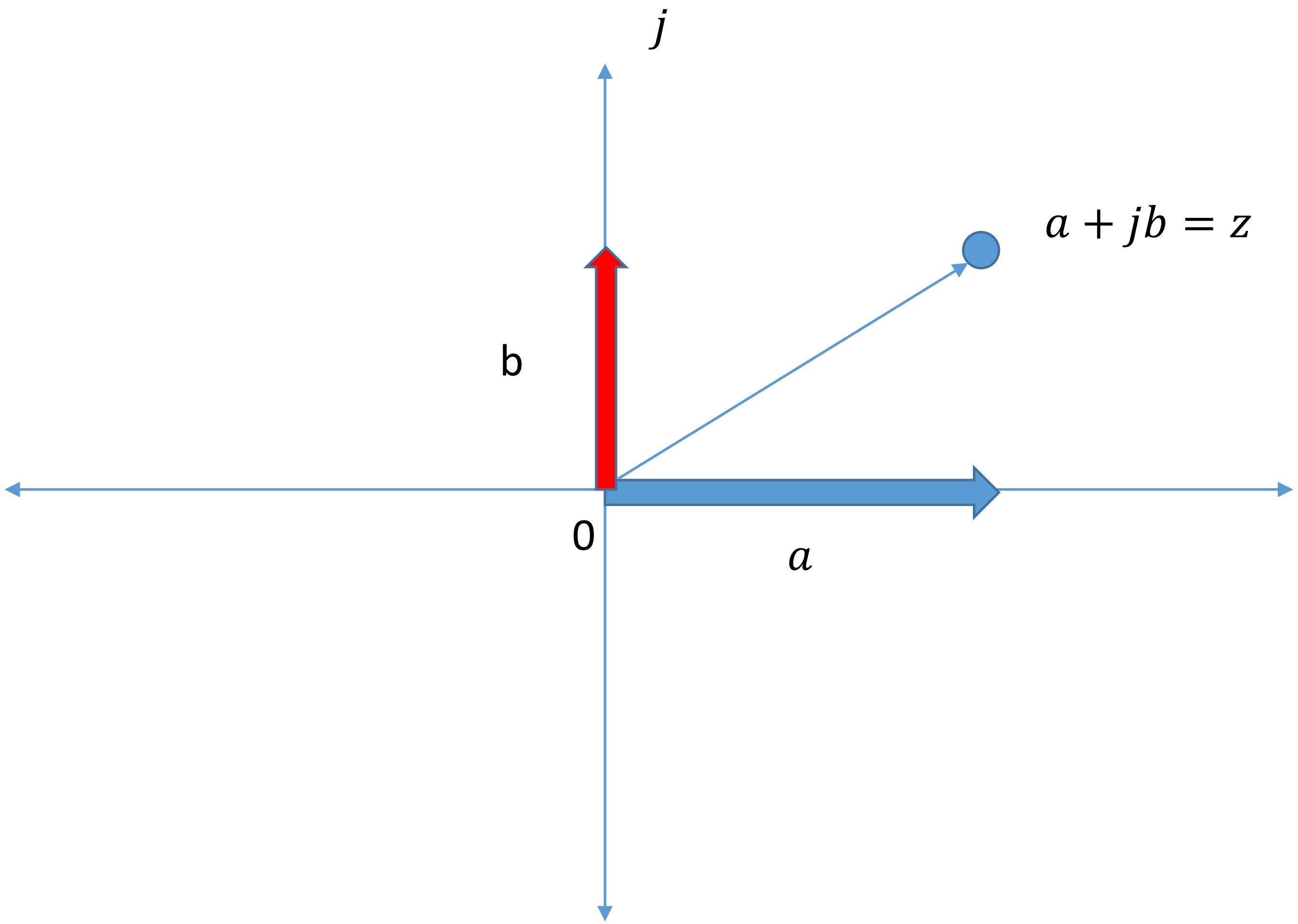
Plotting a Complex Number



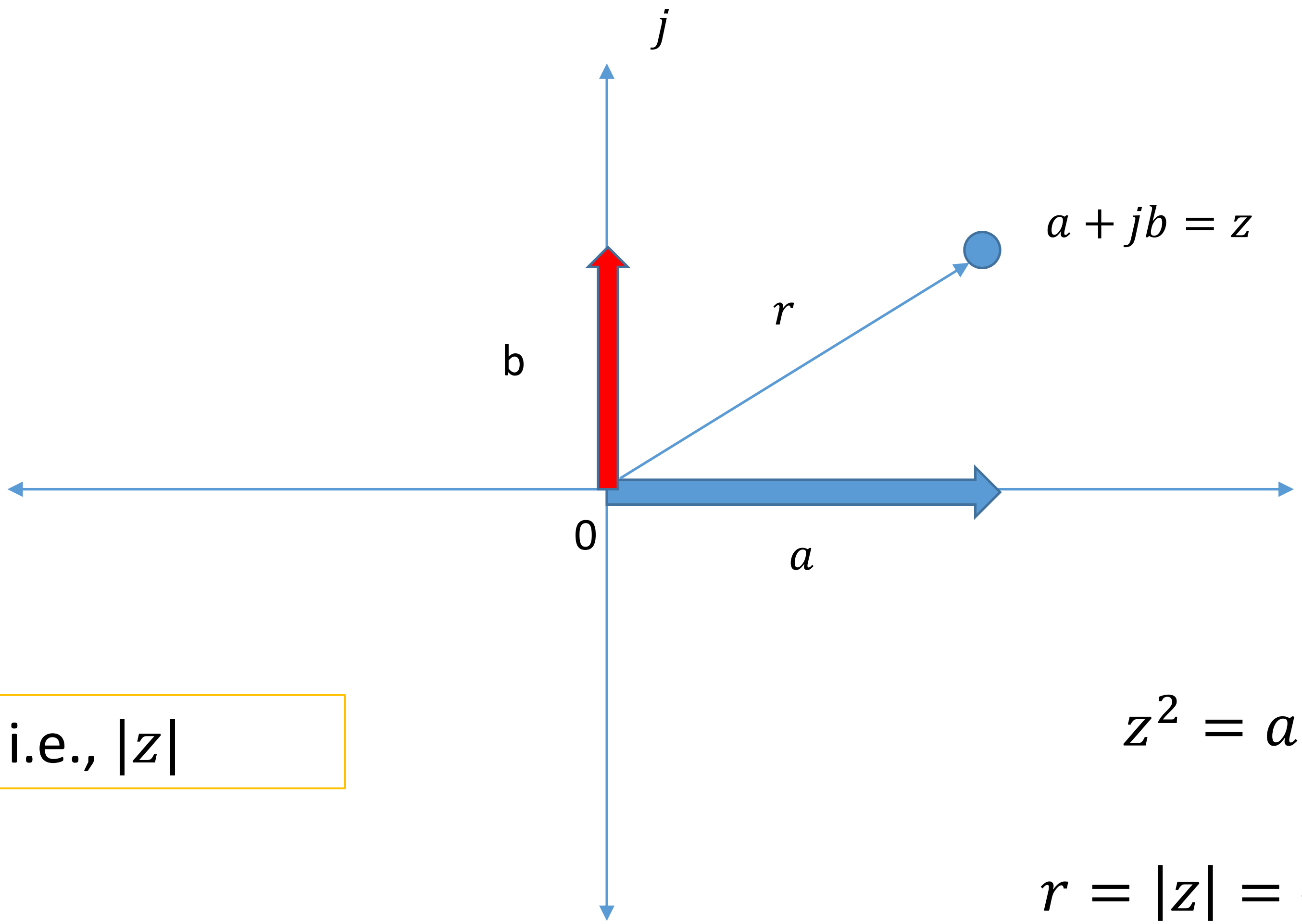
Plotting a Complex Number



Plotting a Complex Number



Q. What is the length of z?

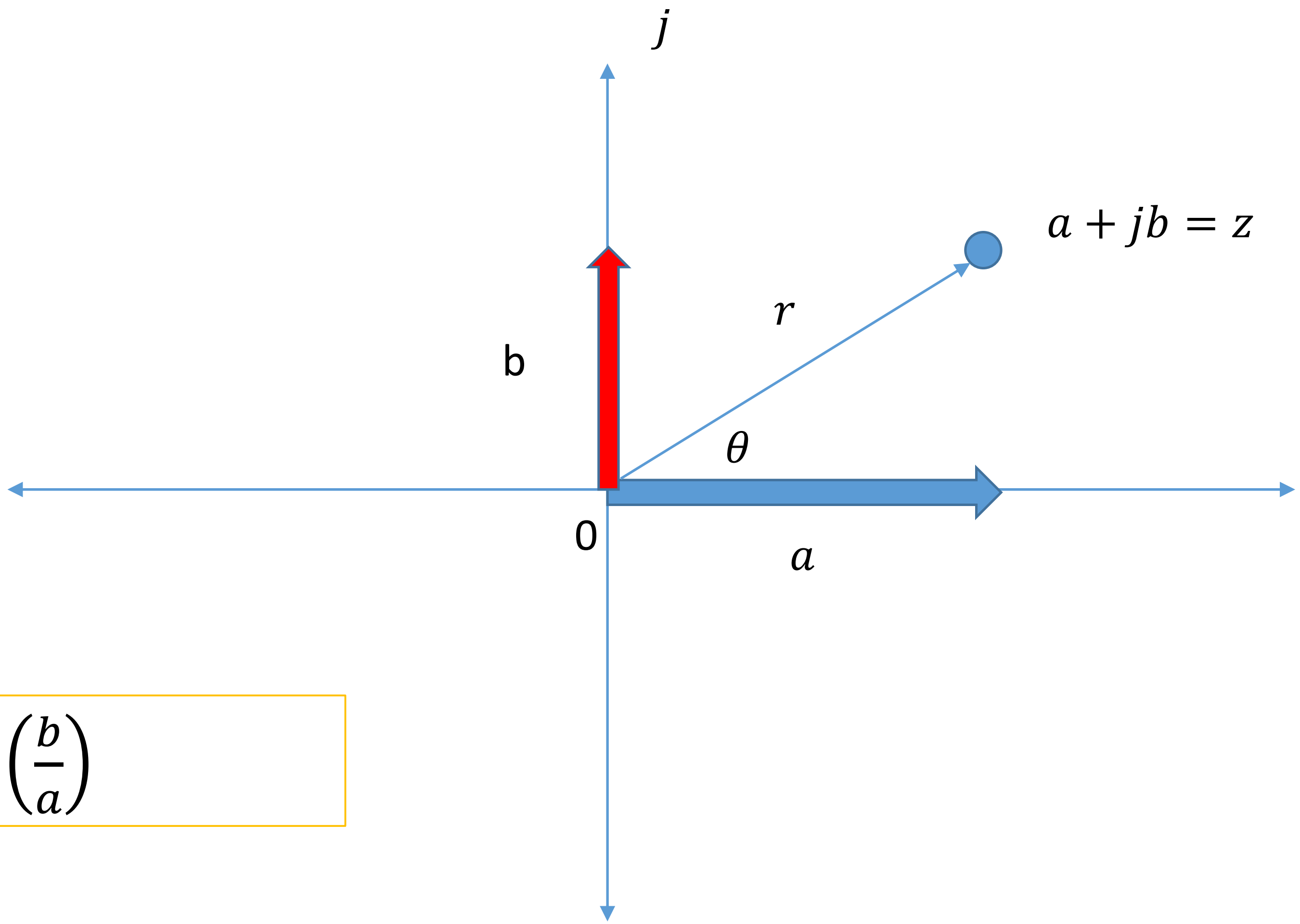


A. Modulus of z, i.e., $|z|$

$$z^2 = a^2 + b^2$$

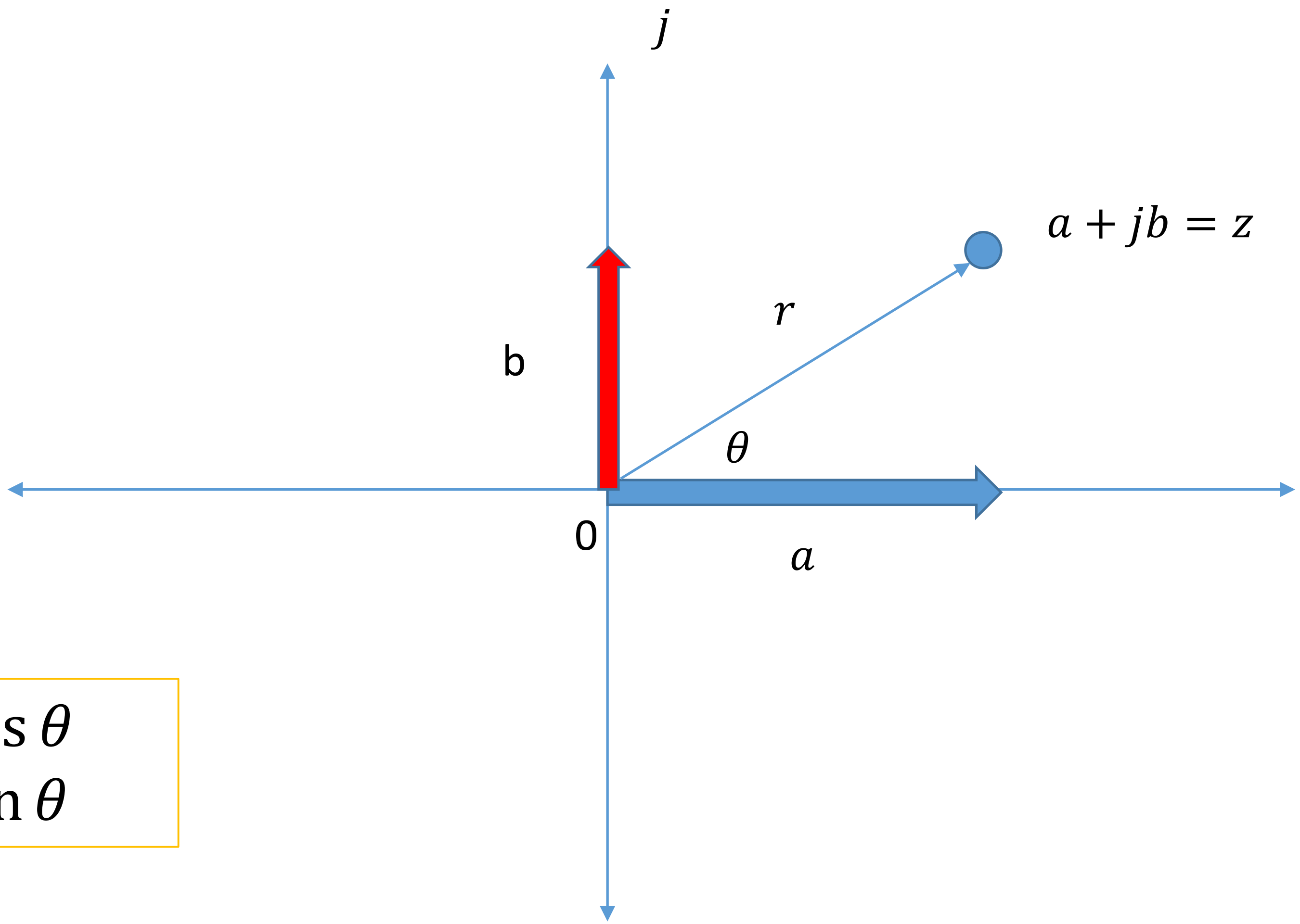
$$r = |z| = \sqrt{a^2 + b^2}$$

Q. What is the angle of z?



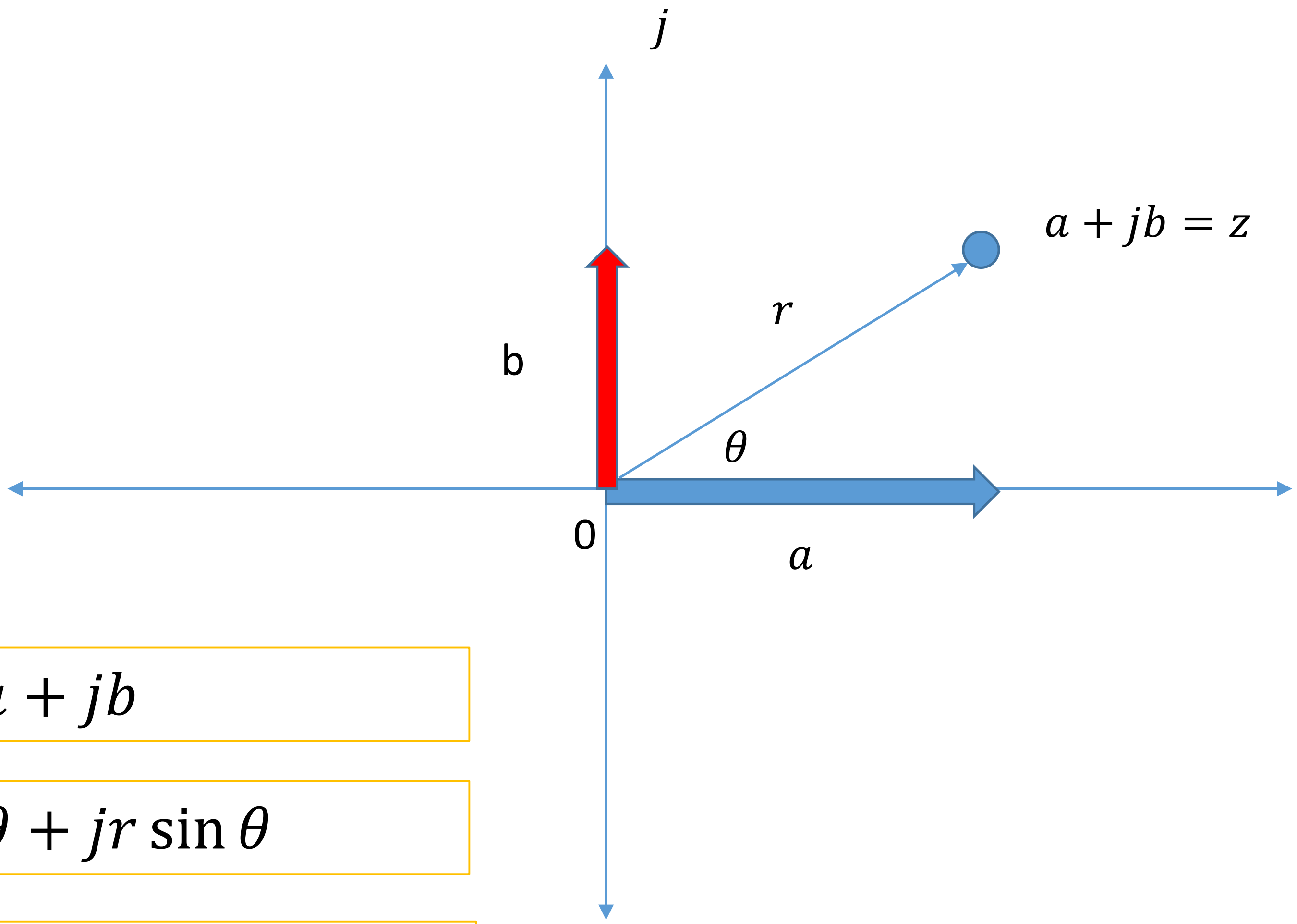
A. $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Q. Can we write a and b in terms of r and θ?



$$a = r\cos\theta$$
$$b = r\sin\theta$$

Q. Can we write z in terms of r and θ?



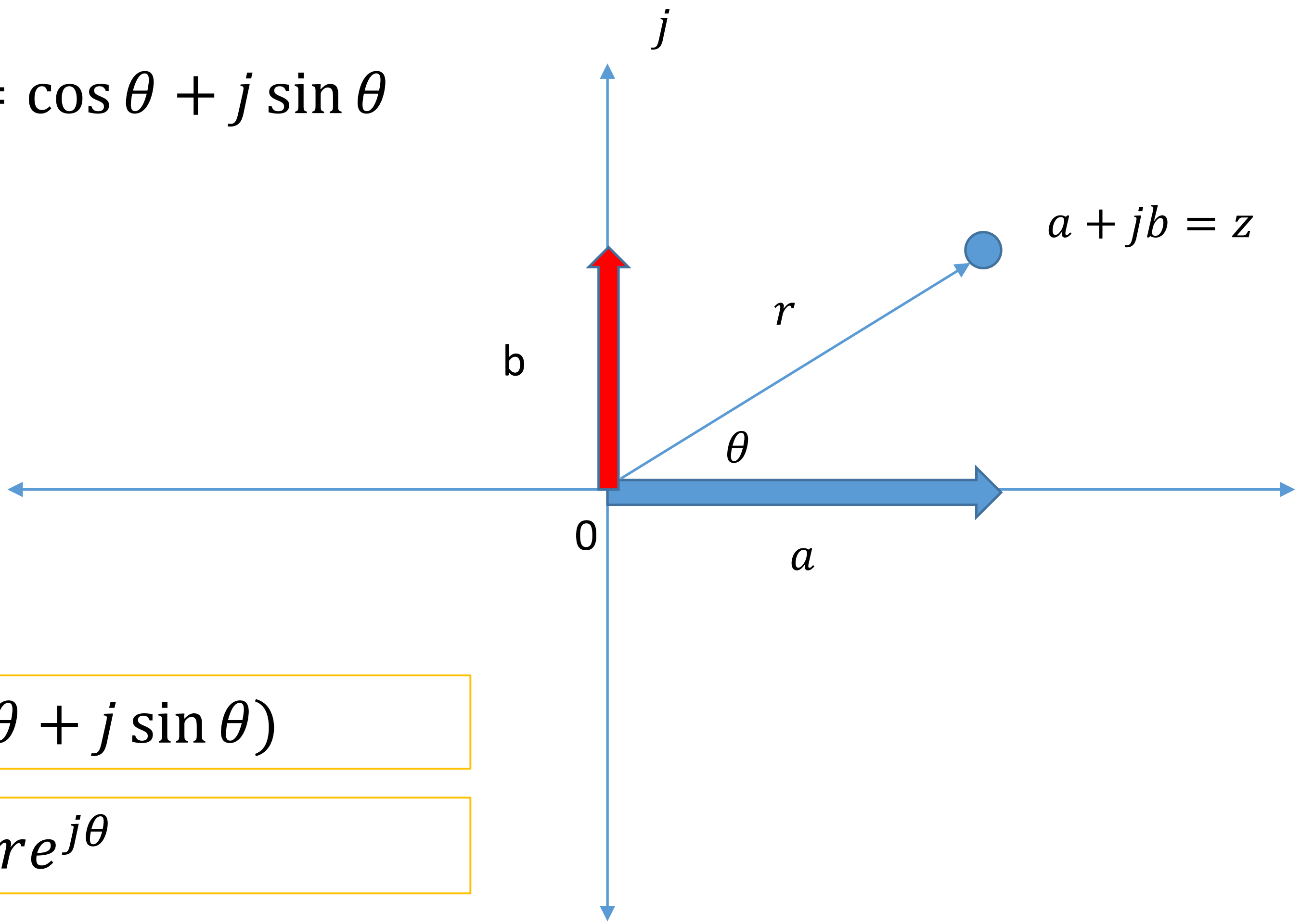
$$z = a + jb$$

$$z = r\cos\theta + jr\sin\theta$$

$$z = r(\cos\theta + j\sin\theta)$$

Q. Can we use Euler's identity to make z look nicer?

$$e^{j\theta} = \cos\theta + j\sin\theta$$

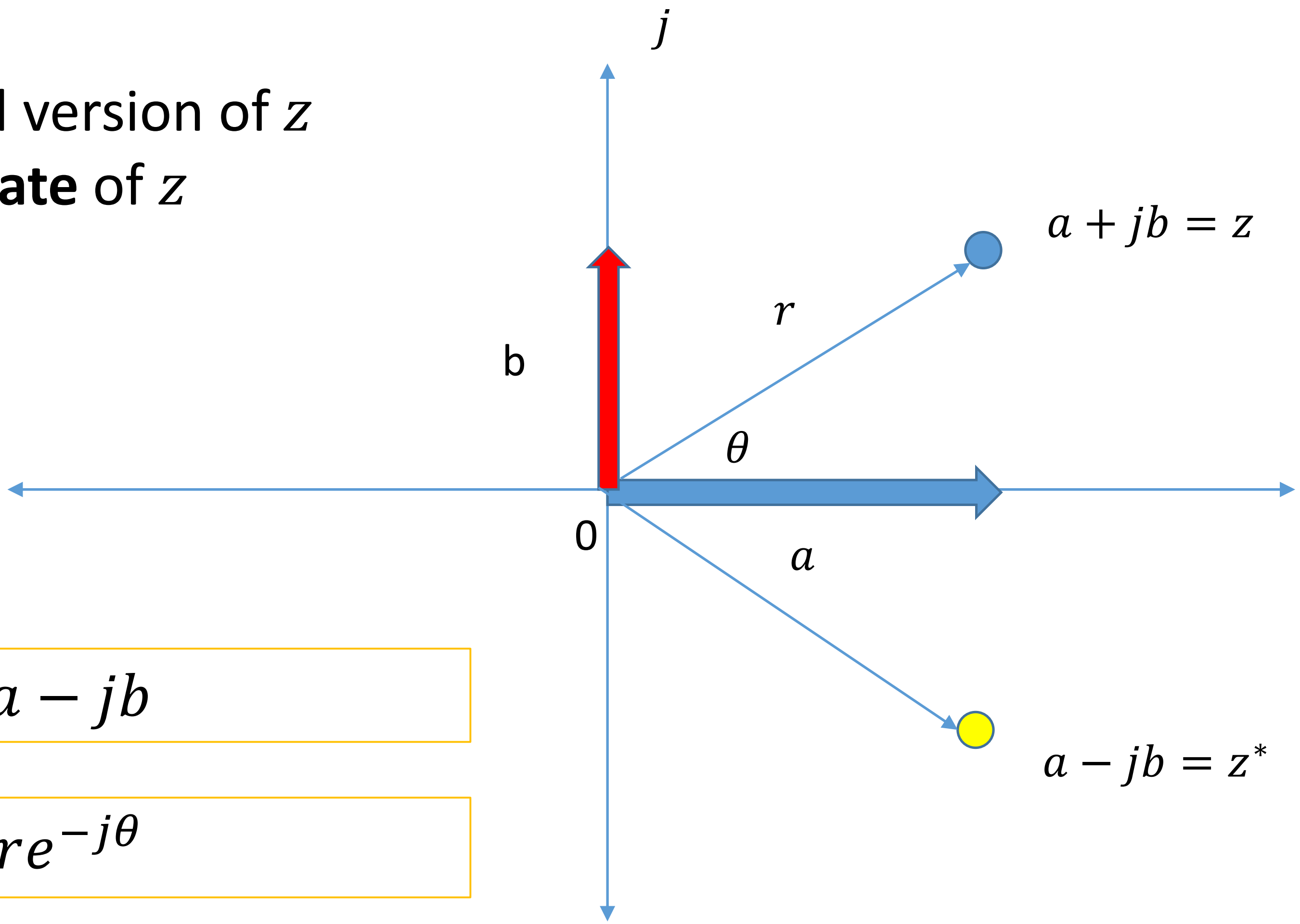


$$z = r(\cos\theta + j\sin\theta)$$

$$z = re^{j\theta}$$

Q. What happens if I change the sign of j?

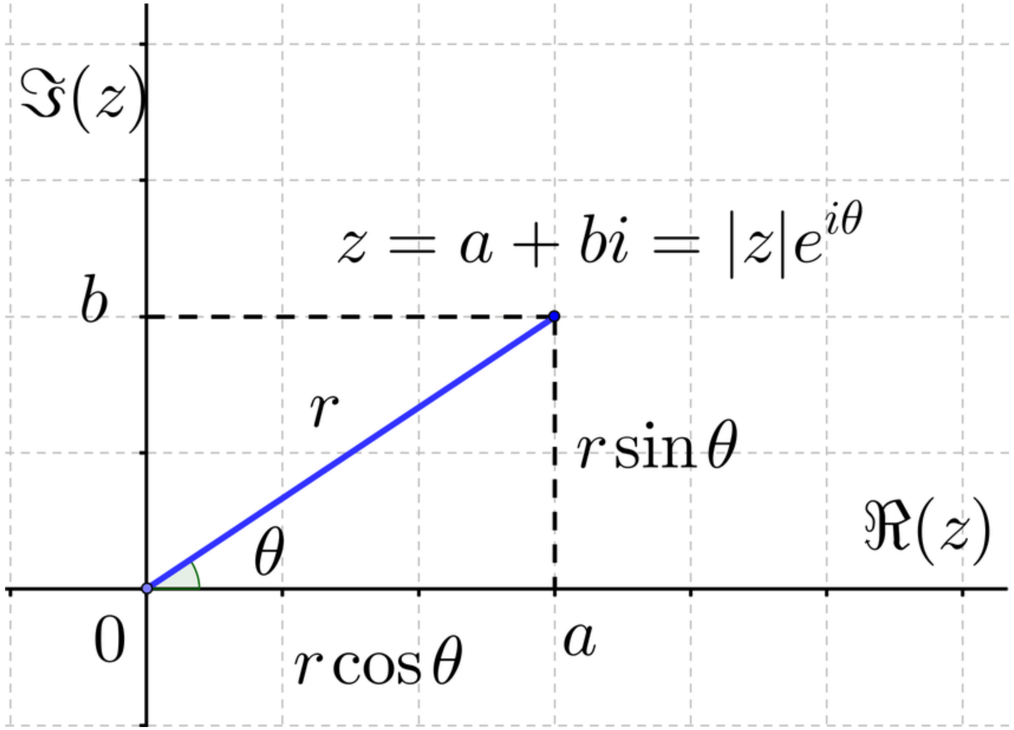
A. You get a flipped version of z that we call **conjugate** of z



$$z^* = a - jb$$

$$z^* = re^{-j\theta}$$

Summary



Cartesian form $\quad z = a + jb$

Polar form $\quad z = |z|e^{j\theta} = re^{j\theta}$

Cartesian to Polar

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\frac{b}{a}$$

Polar to Cartesian

$$a = r\cos\theta$$
$$b = r\sin\theta$$

Some Interesting Results

$$(z^*)^* = z$$

$$zz^* = |z|^2$$

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$e^{j2\pi n} = 1$ (n = integer)

Example

Using both polar and Cartesian forms, determine $z_1 z_2$ and z_1/z_2 for the numbers

$$z_1 = 3 + j4 = 5e^{j53.1^\circ} \qquad \text{and} \qquad z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

Multiplication: Cartesian Form

$$z_1 z_2 = (3 + j4)(2 + j3) = (6 - 12) + j(8 + 9) = -6 + j17$$

Multiplication: Polar Form

$$z_1 z_2 = (5e^{j53.1^\circ})\left(\sqrt{13}e^{j56.3^\circ}\right) = 5\sqrt{13}e^{j109.4^\circ}$$

Division: Cartesian Form

$$\frac{z_1}{z_2} = \frac{3+j4}{2+j3}$$

$$\frac{z_1}{z_2} = \frac{(3+j4)(2-j3)}{(2+j3)(2-j3)} = \frac{18-j1}{2^2+3^2} = \frac{18-j1}{13} = \frac{18}{13} - j\frac{1}{13}$$

Division: Polar Form

$$\frac{z_1}{z_2} = \frac{5e^{j53.1^\circ}}{\sqrt{13}e^{j56.3^\circ}} = \frac{5}{\sqrt{13}}e^{j(53.1^\circ - 56.3^\circ)} = \frac{5}{\sqrt{13}}e^{-j3.2^\circ}$$

Example $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$

(a) $2z_1 - z_2$

$$z_1 = 2e^{j\pi/4} = 2\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = \sqrt{2} + j\sqrt{2}$$

$$z_2 = 8e^{j\pi/3} = 8\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 4 + j4\sqrt{3}$$

$$2z_1 - z_2 = 2(\sqrt{2} + j\sqrt{2}) - (4 + j4\sqrt{3}) = (2\sqrt{2} - 4) + j(2\sqrt{2} - 4\sqrt{3}) = -1.17 - j4.1$$

Example $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$

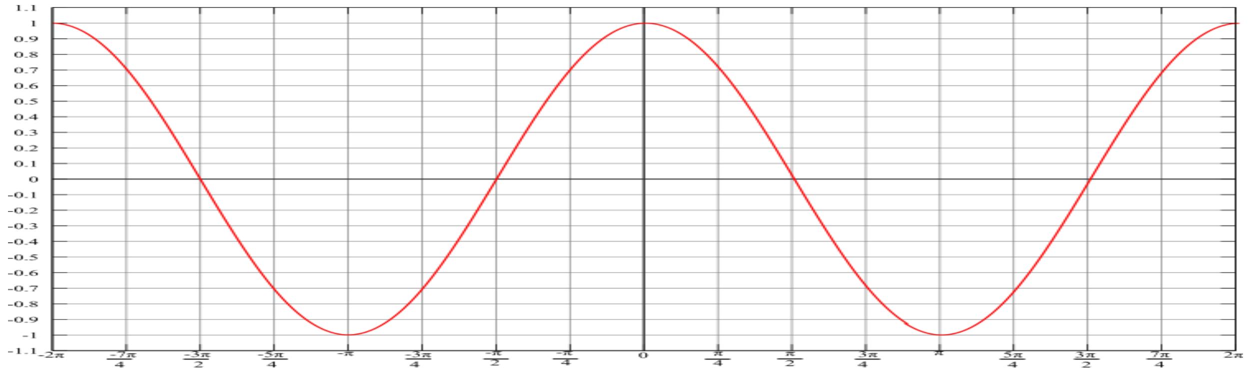
(c) z_1/z_2^2

$$\frac{z_1}{z_2^2} = \frac{2e^{j\pi/4}}{(8e^{j\pi/3})^2} = \frac{2e^{j\pi/4}}{64e^{j2\pi/3}} = \frac{1}{32}e^{j(\pi/4-2\pi/3)} = \frac{1}{32}e^{-j(5\pi/12)}$$

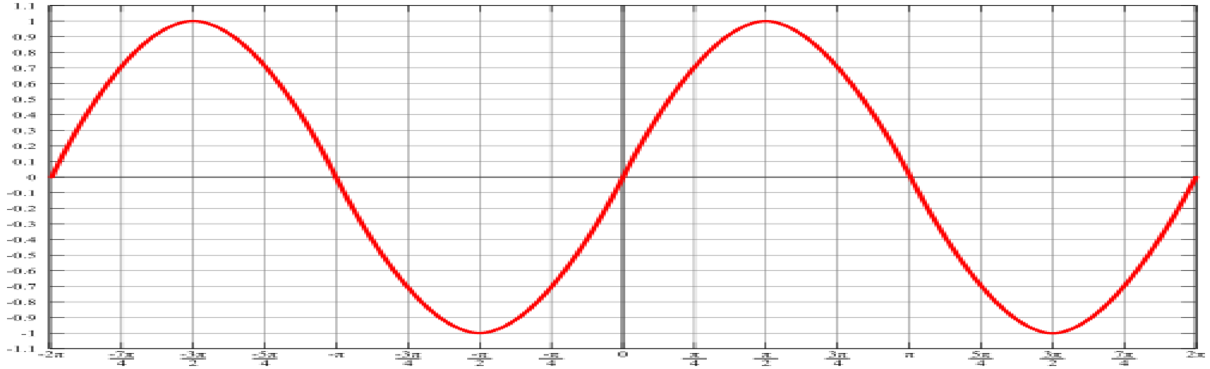
A Sine and a Cosine Walk Into an Imaginary Bar…

The Complex Sinusoid - $e^{j\omega t}$

$\cos(\omega t)$

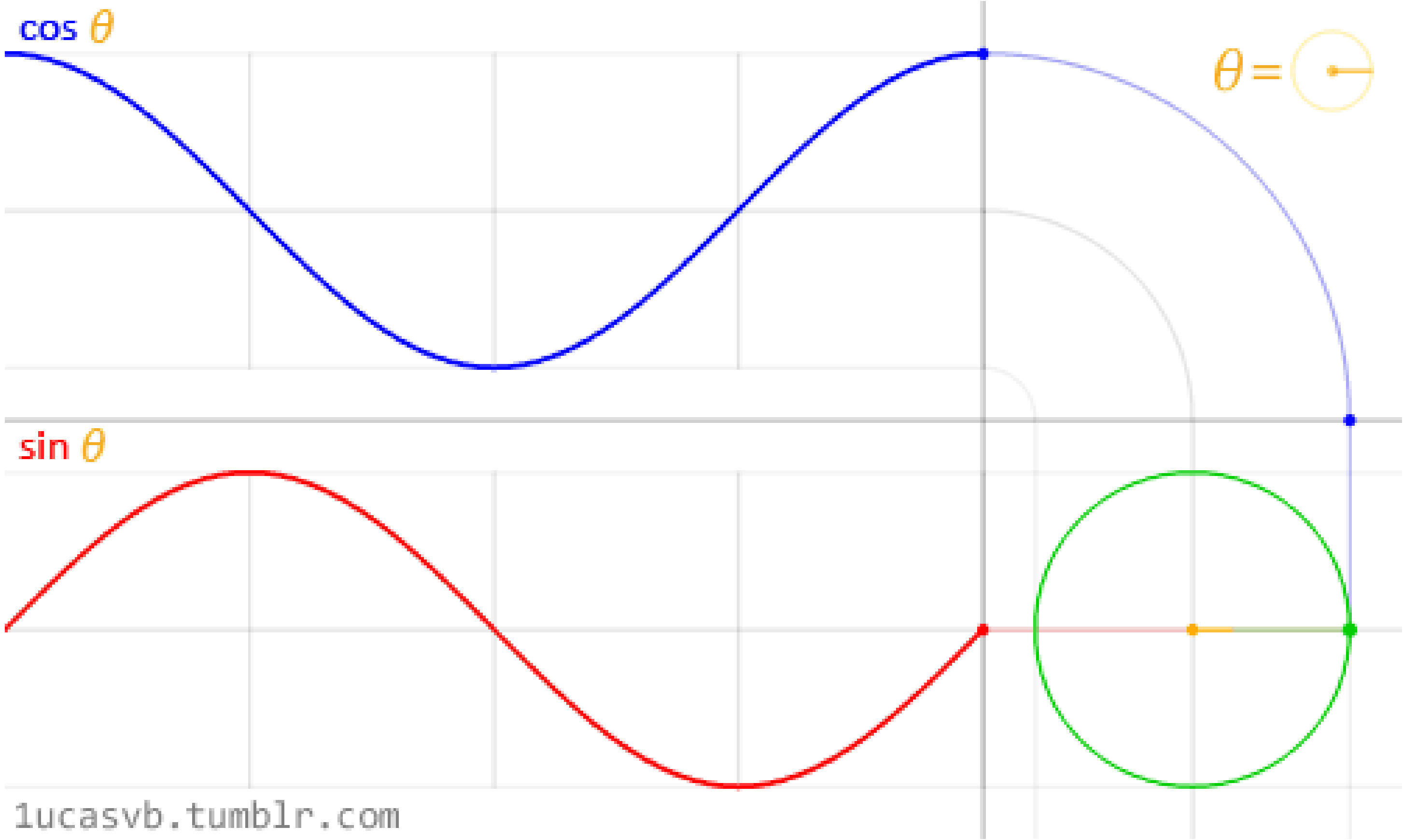


$\sin(\omega t)$

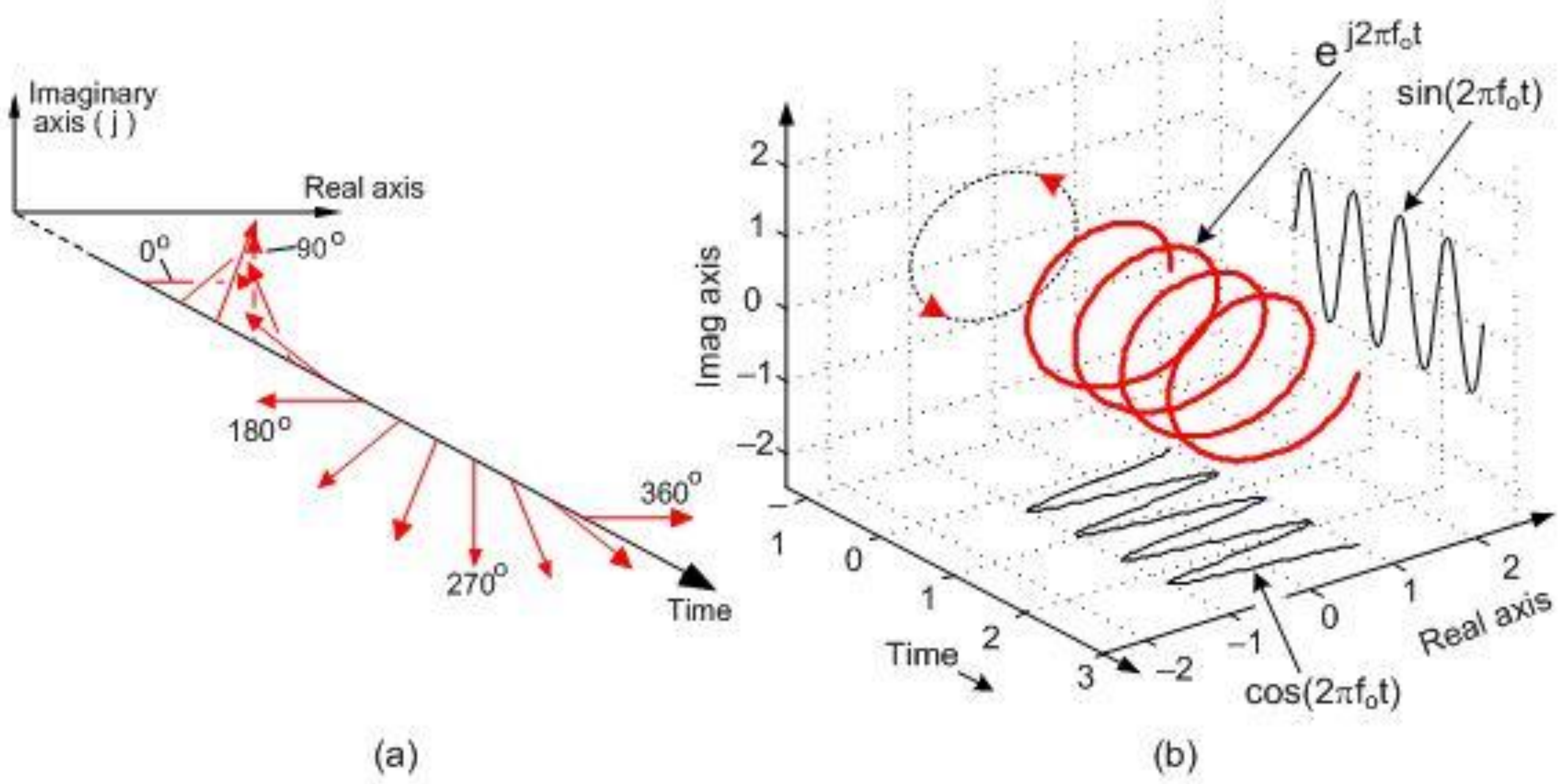


The Complex Sinusoid - $e^{j\omega t}$

$$\cos(\omega t) + j\sin(\omega t)$$



The Complex Sinusoid - $e^{j\omega t}$ $= \cos(\omega t) + j\sin(\omega t)$



The Complex Exponential - e^{st}

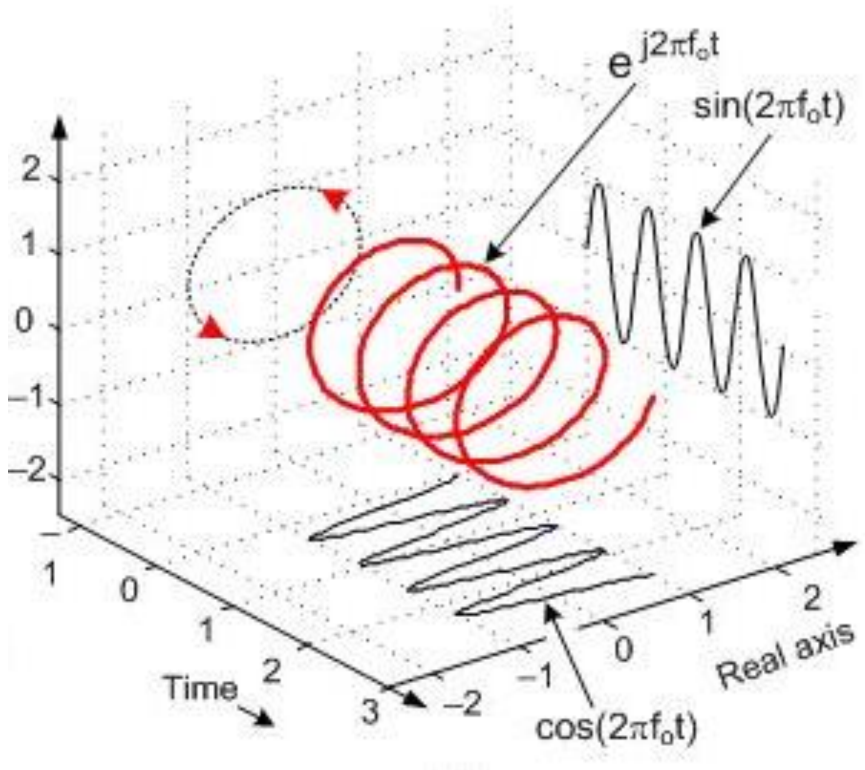
$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t} e^{j\omega t}$$

Exponential
decay or growth

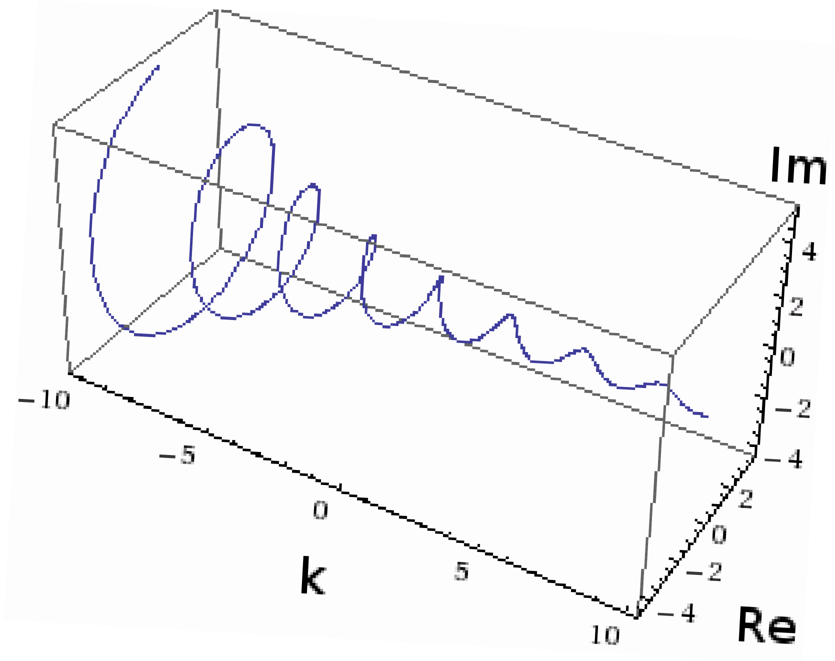
Complex sinusoid

The Complex Exponential: $e^{st} = e^{\sigma t}\, e^{j\omega t}$

$\sigma = 0$



$\sigma < 0$



Example

Show that the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

is periodic and that its fundamental period is $2\pi/\omega_0$.

$x(t)$ will be periodic if $\quad e^{j\omega_0(t+T)} = e^{j\omega_0 t}$

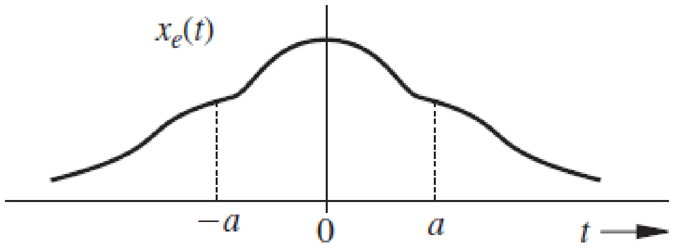
$$e^{j\omega_0(t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \Rightarrow e^{j\omega_0 T} = 1$$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T.

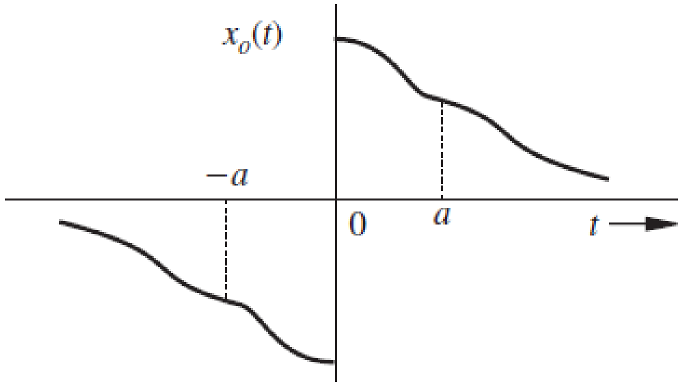
If $\omega_0 \neq 0$ $\quad \boxed{\omega_0 T = m2\pi \quad \text{or} \quad T = m\frac{2\pi}{\omega_0} \quad m = \text{positive integer}}$

Smallest value of T occurs at $m = 1$, and we call it the fundamental time-period $T_0 = \frac{2\pi}{\omega_0}$

Even vs Odd Signals



(a)



(b)

$$x_e(t) = x_e(-t)$$

Even function – symmetric about the vertical axis.

$$x_o(t) = -x_o(-t)$$

Odd function – anti-symmetric about the vertical axis.

Products of Even and Odd Functions

$$\text{even function} \times \text{odd function} = \text{odd function}$$
$$\text{odd function} \times \text{odd function} = \text{even function}$$
$$\text{even function} \times \text{even function} = \text{even function}$$

Products of Even and Odd Functions

Let $x(t) = x_1(t)x_2(t)$. If $x_1(t)$ and $x_2(t)$ are both even, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)x_2(t) = x(t)$$

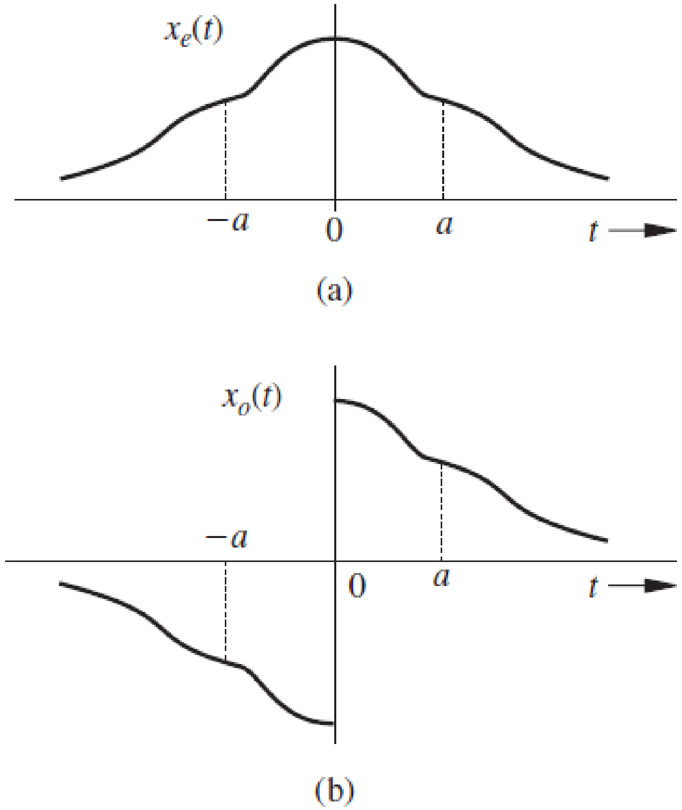
and $x(t)$ is even. If $x_1(t)$ and $x_2(t)$ are both odd, then

$$x(-t) = x_1(-t)x_2(-t) = -x_1(t)\,[-x_2(t)] = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ is even and $x_2(t)$ is odd, then

$$x(-t) = x_1(-t)\,x_2(-t) = x_1(t)\,[-x_2(t)] = -x_1(t)x_2(t) = -x(t)$$

Integrals of Even and Odd Functions



$$\int_{-a}^{a} x_e(t)\,dt = 2\int_{0}^{a} x_e(t)\,dt$$

$$\int_{-a}^{a} x_o(t)\,dt = 0$$

Writing a signal as sum of Even and Odd

$$x(t) = \underbrace{\tfrac{1}{2}[x(t) + x(-t)]}_{\text{even}} + \underbrace{\tfrac{1}{2}[x(t) - x(-t)]}_{\text{odd}}$$

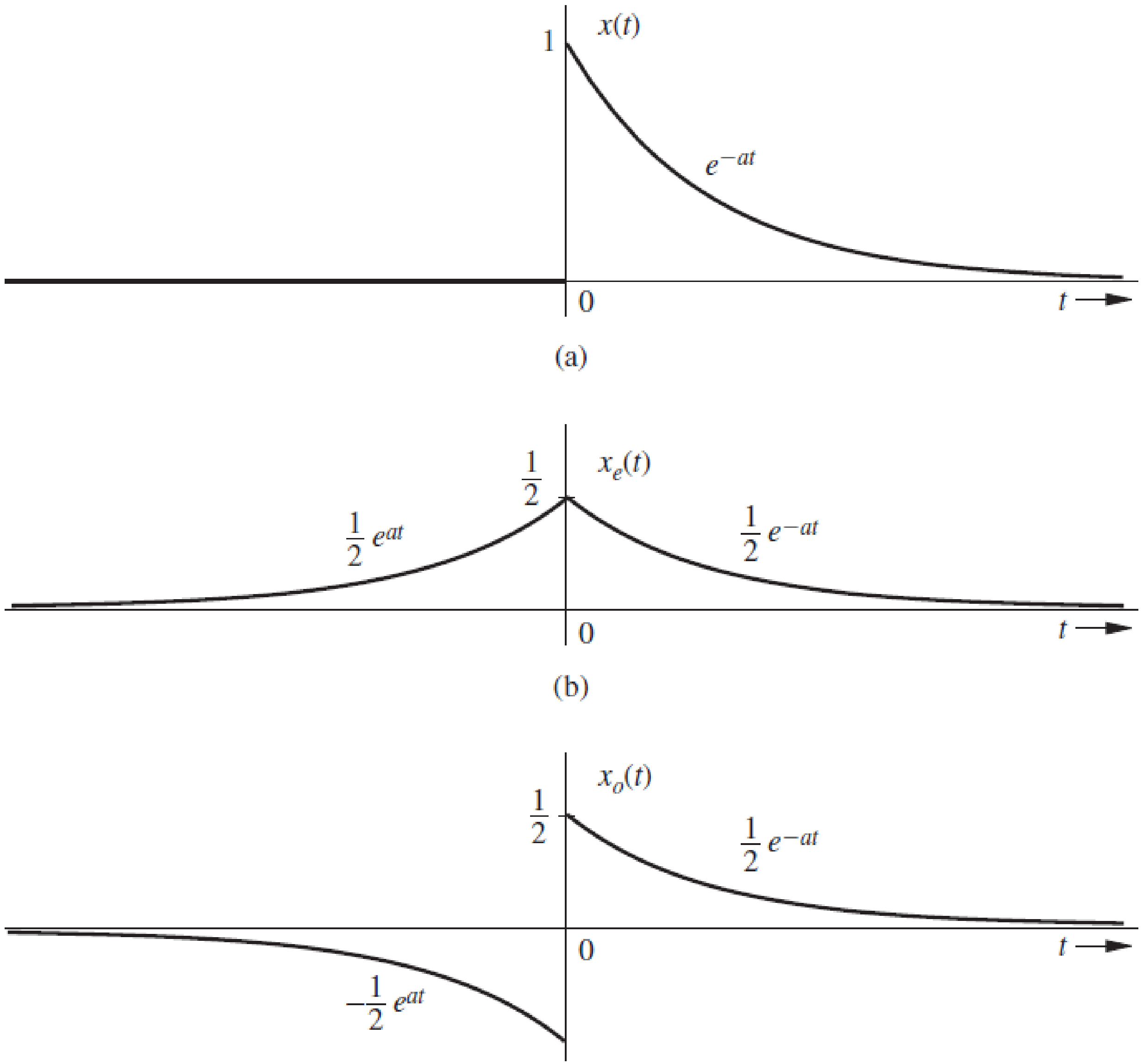
Examples

Find and sketch the even and odd components of $x(t) = e^{-at}u(t)$.

$$x(t) = \underbrace{\tfrac{1}{2}[x(t) + x(-t)]}_{\text{even}} + \underbrace{\tfrac{1}{2}[x(t) - x(-t)]}_{\text{odd}}$$

$$x_e(t) = \tfrac{1}{2}[e^{-at}u(t) + e^{at}u(-t)]$$

$$x_o(t) = \tfrac{1}{2}[e^{-at}u(t) - e^{at}u(-t)]$$


$x(t)$
1
e^{-at}
0
t
(a)
$x_e(t)$
$\frac{1}{2}$
$\frac{1}{2}e^{at}$
$\frac{1}{2}e^{-at}$
0
t
(b)
$x_o(t)$
$\frac{1}{2}$
$\frac{1}{2}e^{-at}$
$-\frac{1}{2}e^{at}$
0
t

Find the even and odd components of e^{jt}.

$$x_e(t) = \frac{1}{2}[e^{jt} + e^{-jt}] = \cos t$$

$$x_o(t) = \frac{1}{2}[e^{jt} - e^{-jt}] = j\sin t$$

Questions?? Thoughts??

ES 332
Signals and Systems

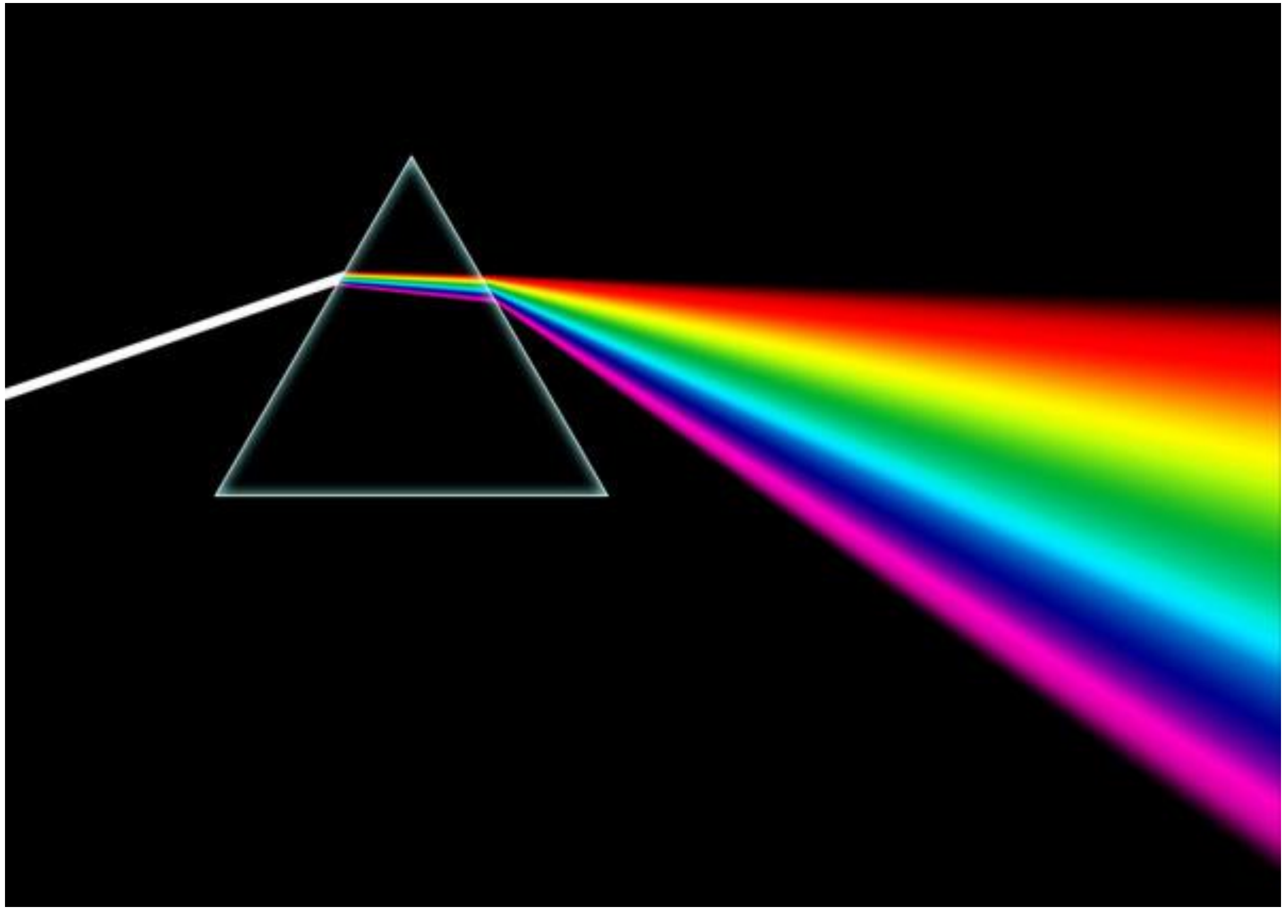
with

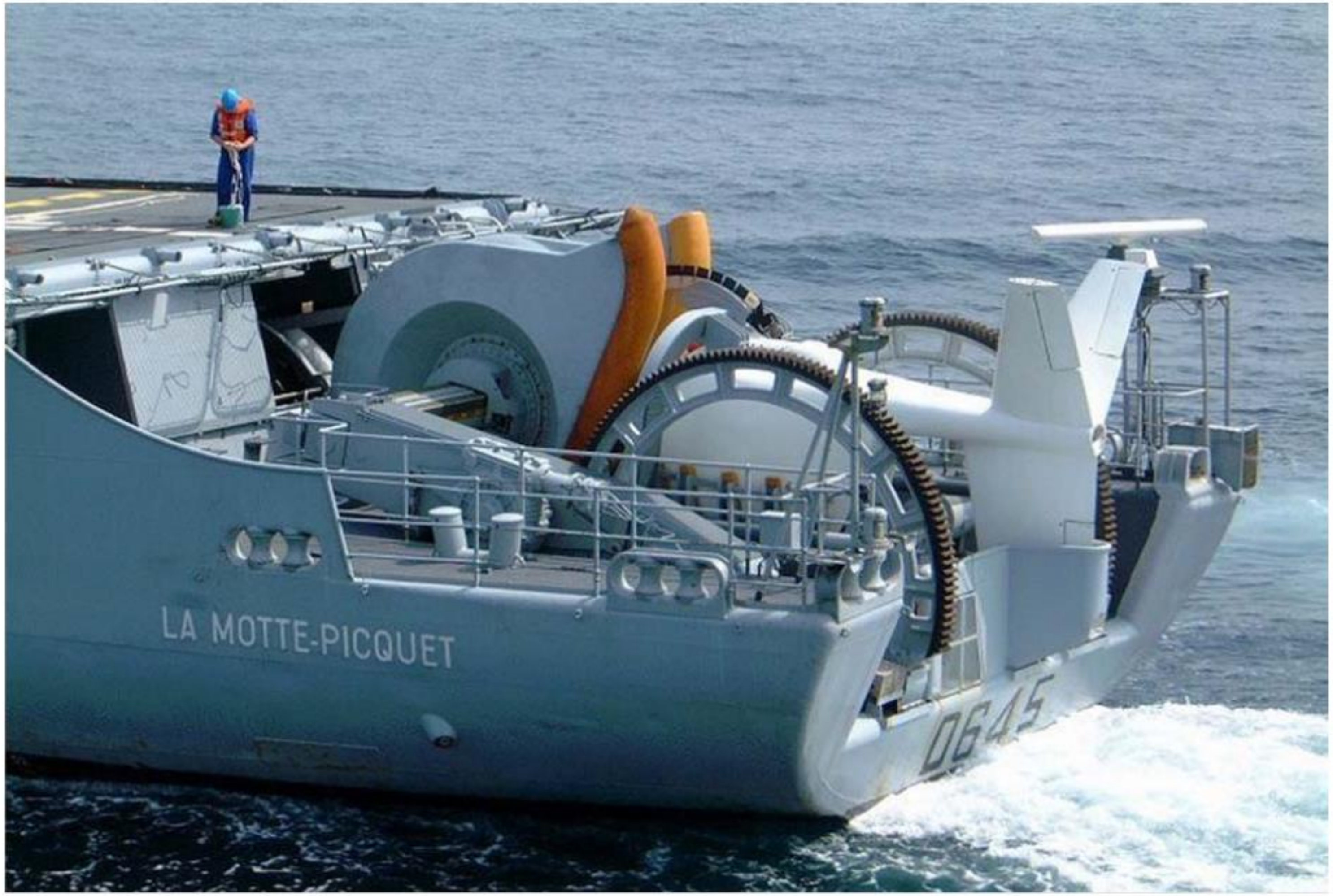
Dr. Naveed R. Butt

@

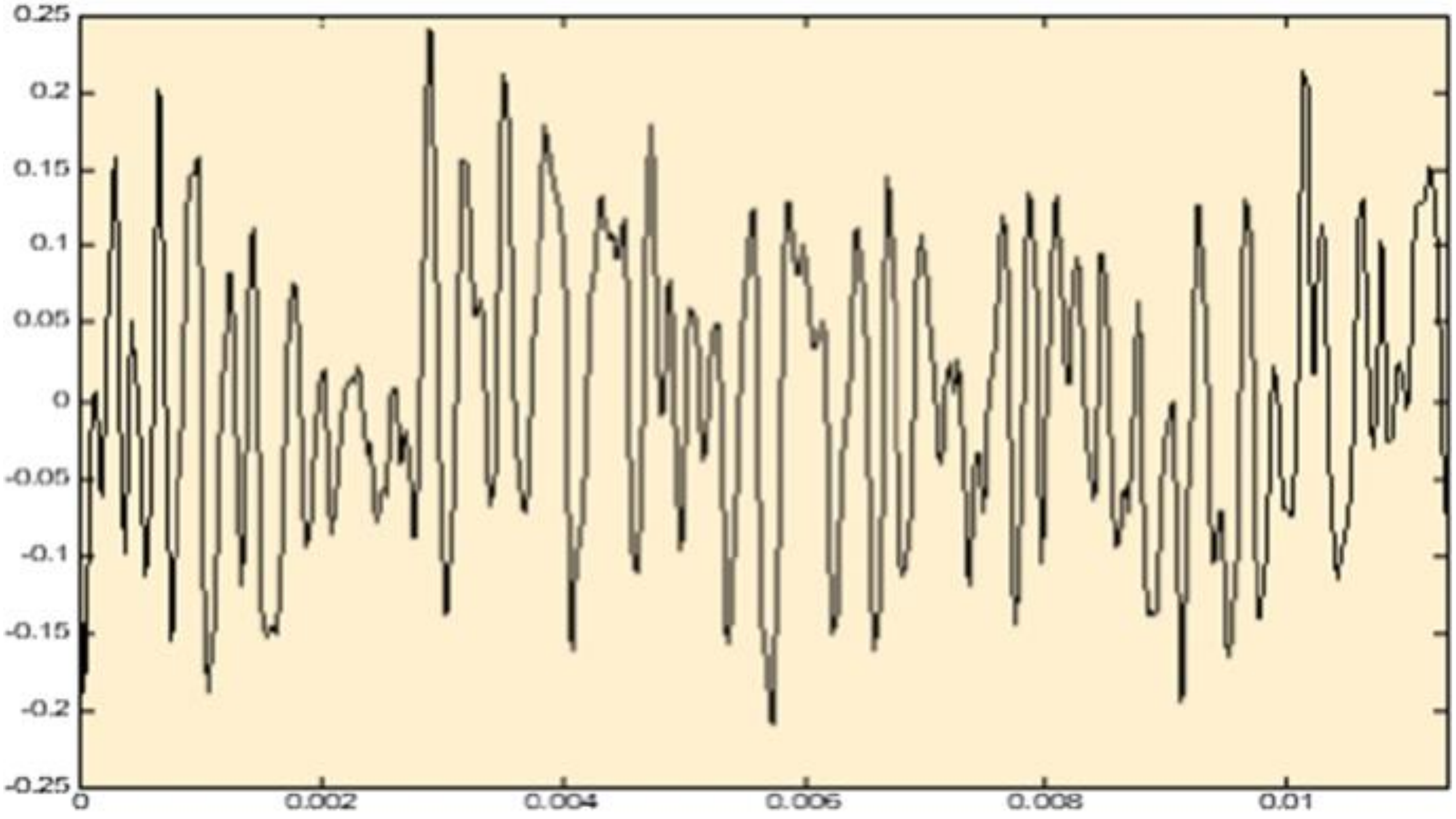
GIKI - FES

Spectra – *the Ghosts in Your Signal*

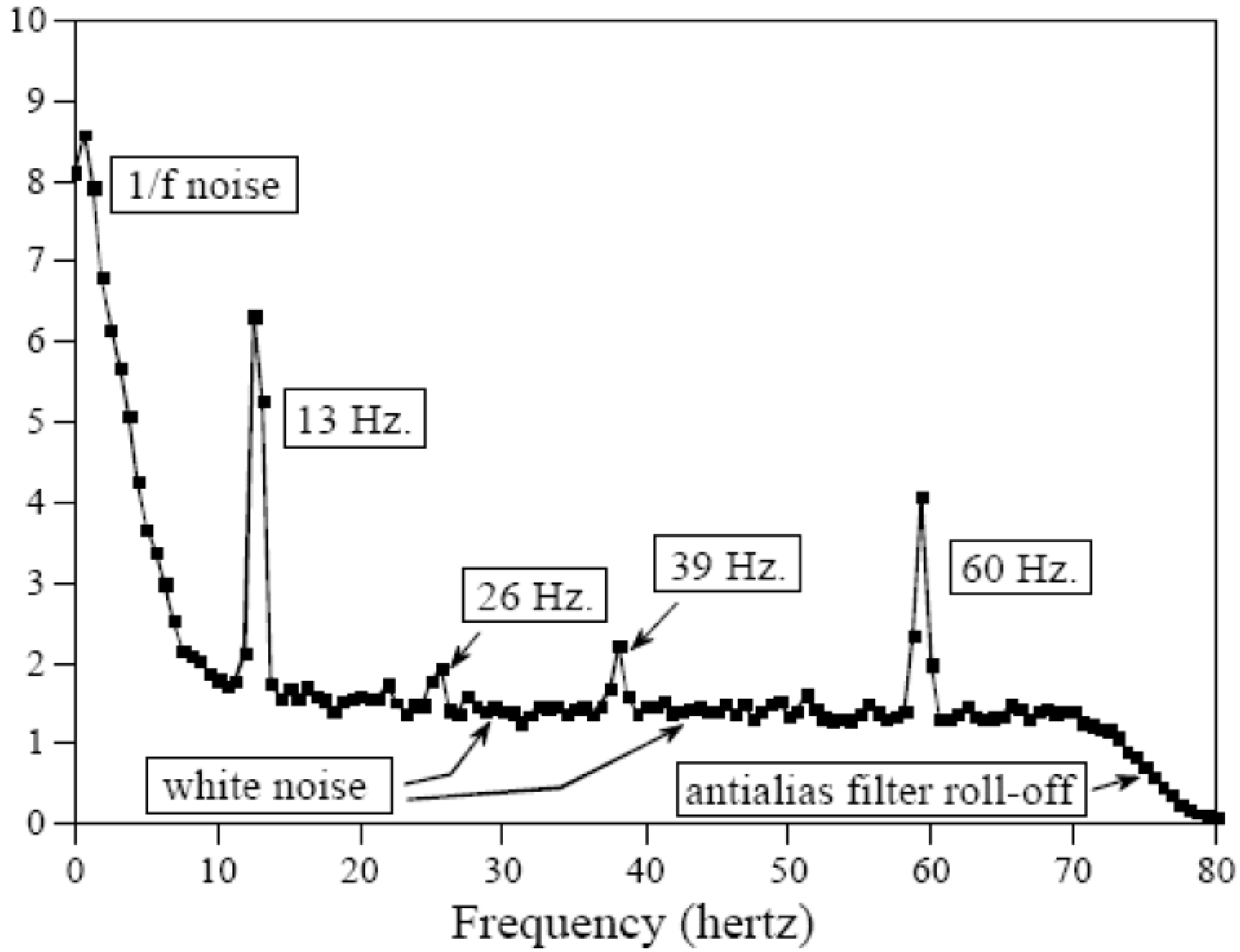




Sonar


0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
0.002
0.004
0.006
0.008
0.01

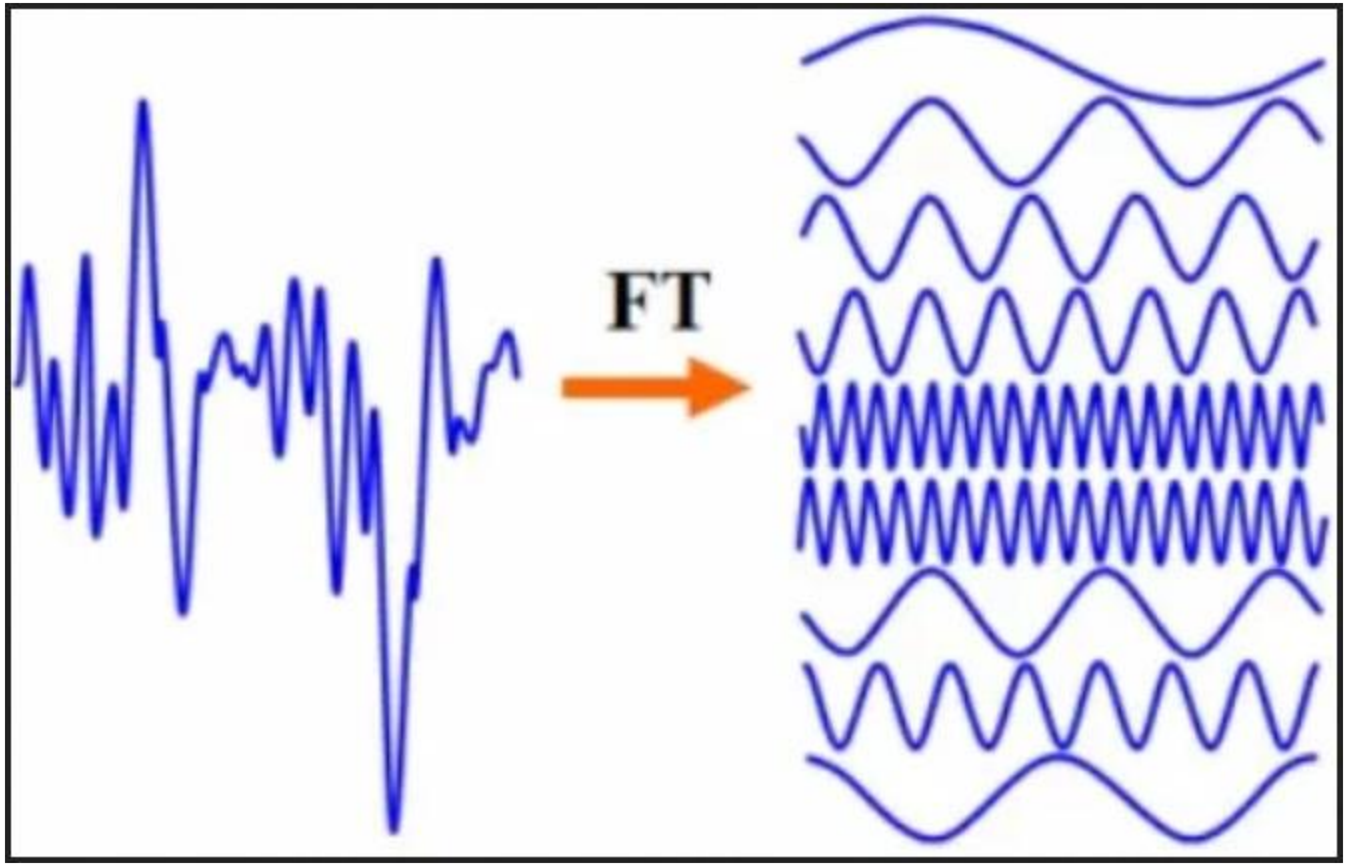

J. Boilly Del.
Geille Sculp.


10
9
8
7
6
5
4
3
2
1
0
1/f noise
13 Hz.
26 Hz.
39 Hz.
60 Hz.
white noise
antialias filter roll-off
0
10
20
30
40
50
60
70
80
Frequency (hertz)

Q. Can we write signals as sums of periodic functions (frequencies)?

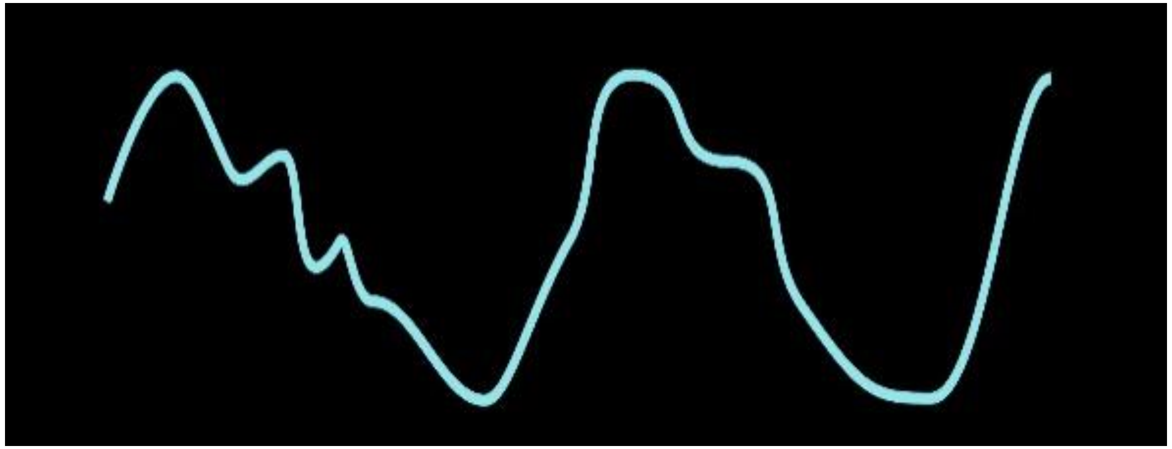
This is exactly what Fourier Transform does – it tries to write every signal as a sum of sinusoids.

Q. Can we write signals as sums of periodic functions (frequencies)?

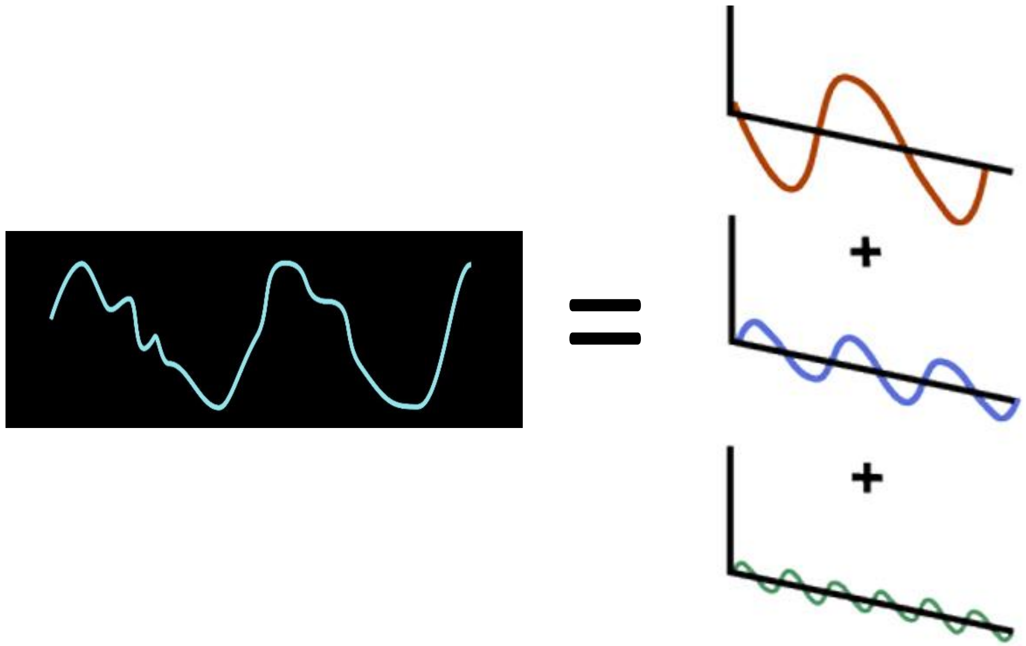


Baking a Fourier Cake

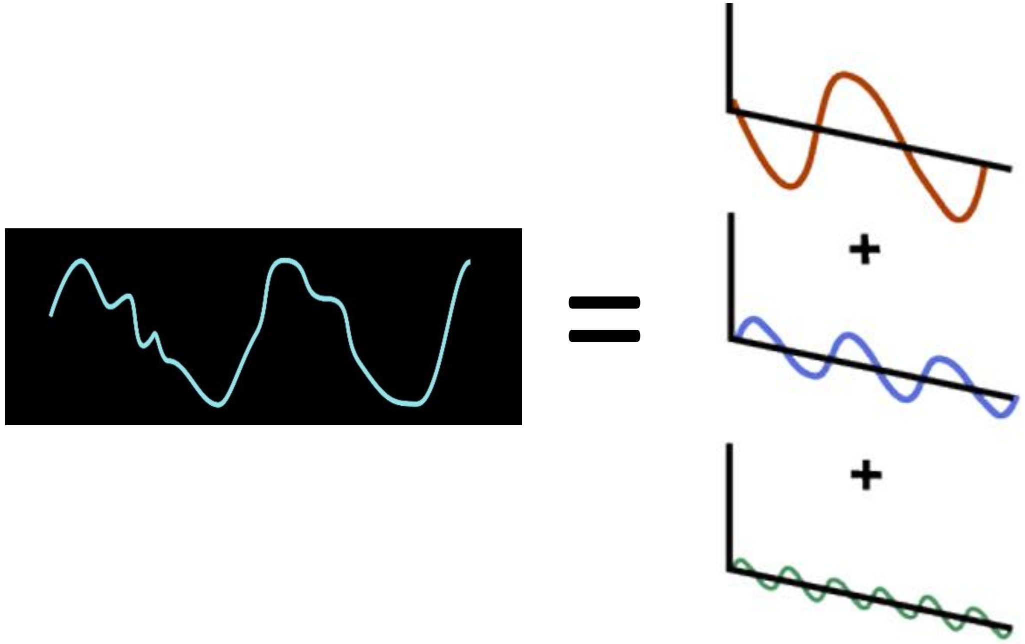
- **Given**: Signal shape (time-domain)
- **Ingredients**: Sinusoids of different frequencies
- **Choose**: How much of the each ingredient (sinusoid) to use?



- In Fourier Transform, we want to look at signals in terms of a fixed set of ingredients
  - Ingredients : Sinusoids of different frequencies

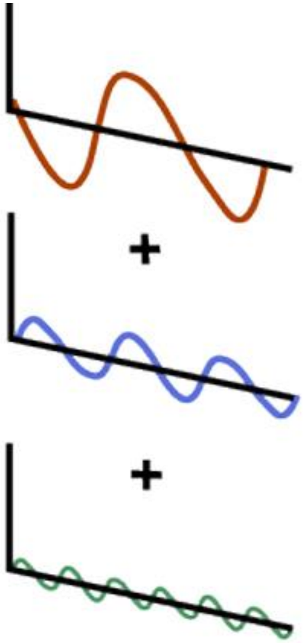


- In Fourier Transform, we want to look at signals in terms of a fixed set of ingredients
  - Ingredients : Sinusoids of different frequencies

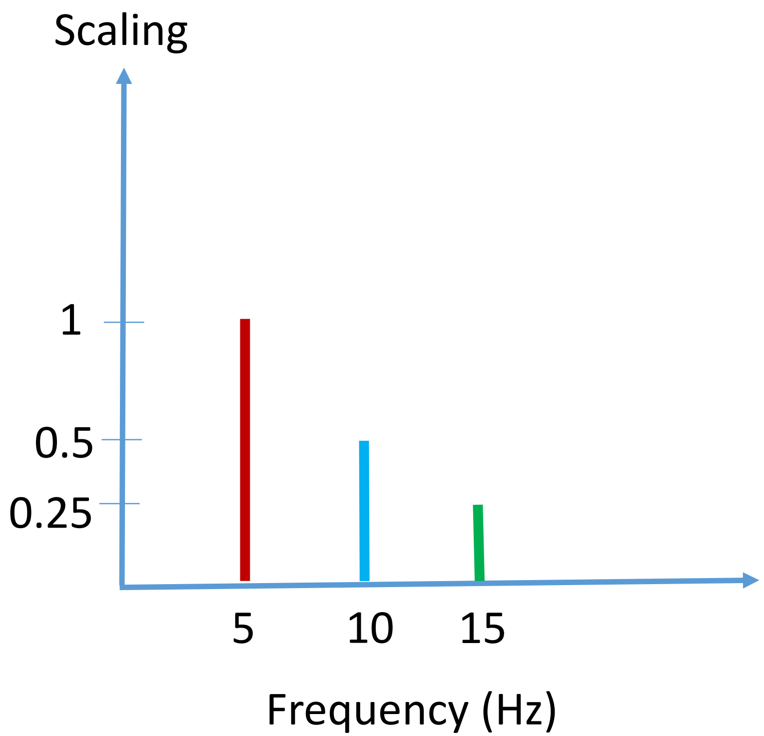


Ingredient (sinusoid frequency)	Amount (scaling)	Process
f_1	1	**Add all**
f_2	0.5	
f_3	0.25	

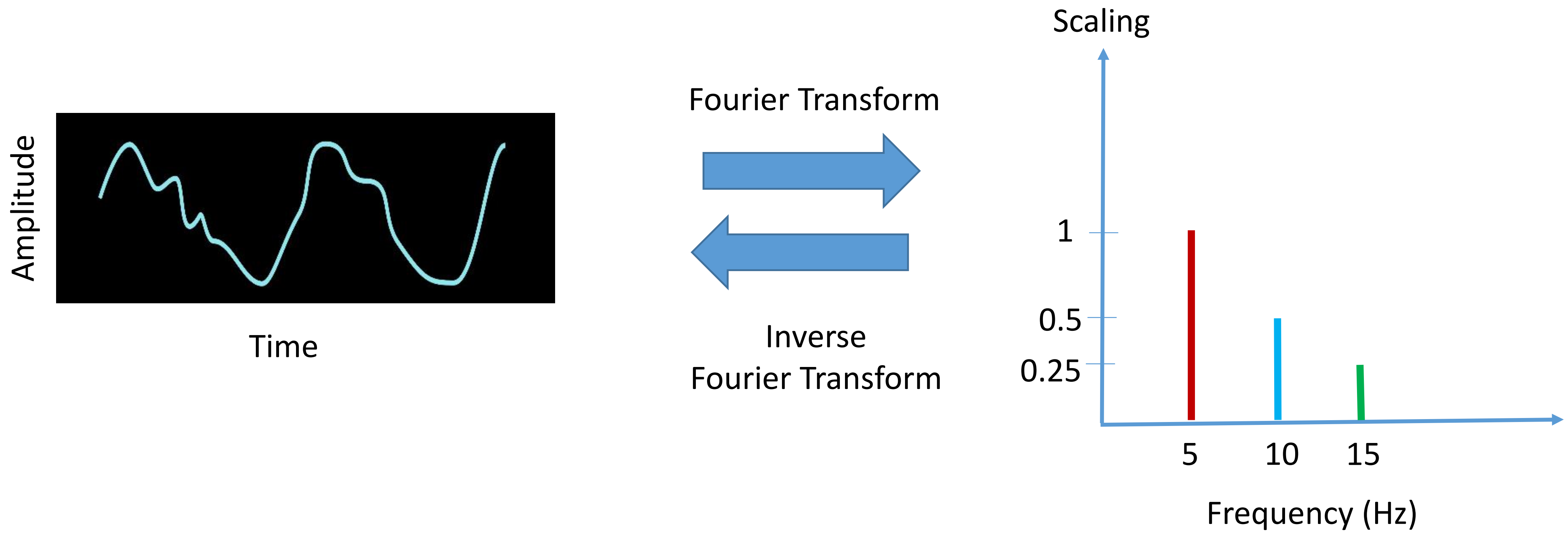
- How is this shown after Fourier transform?

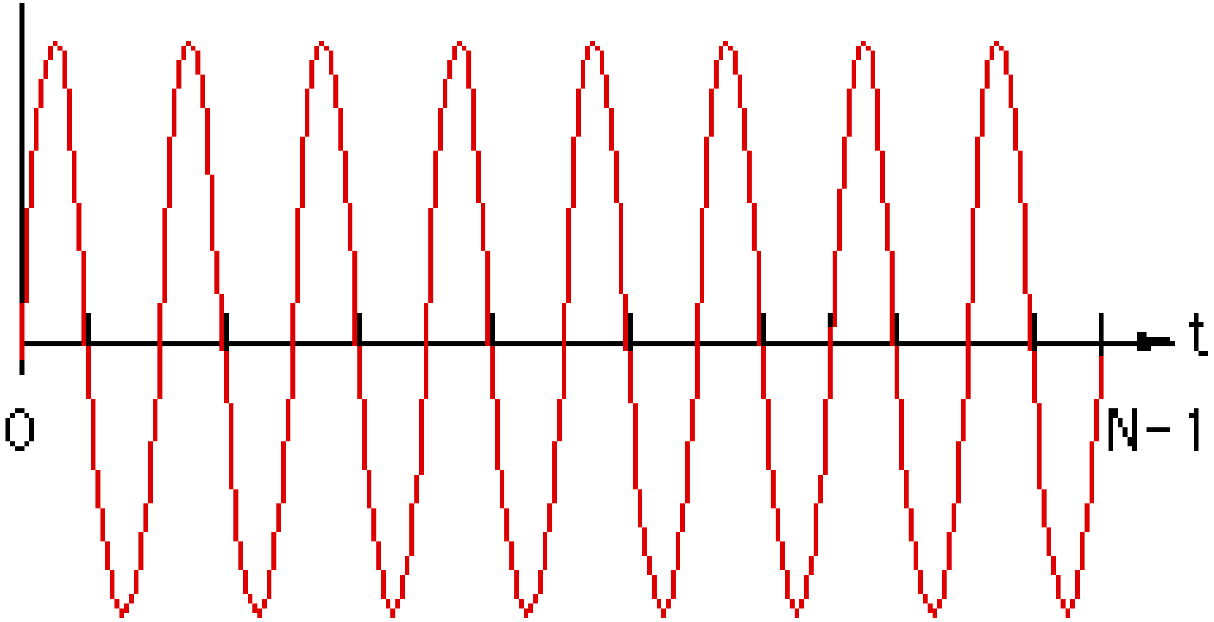


Ingredient (sinusoid frequency)	Amount (scaling)	Process
5 Hz	1	**Add all**
10 Hz	0.5	
15 Hz	0.25	



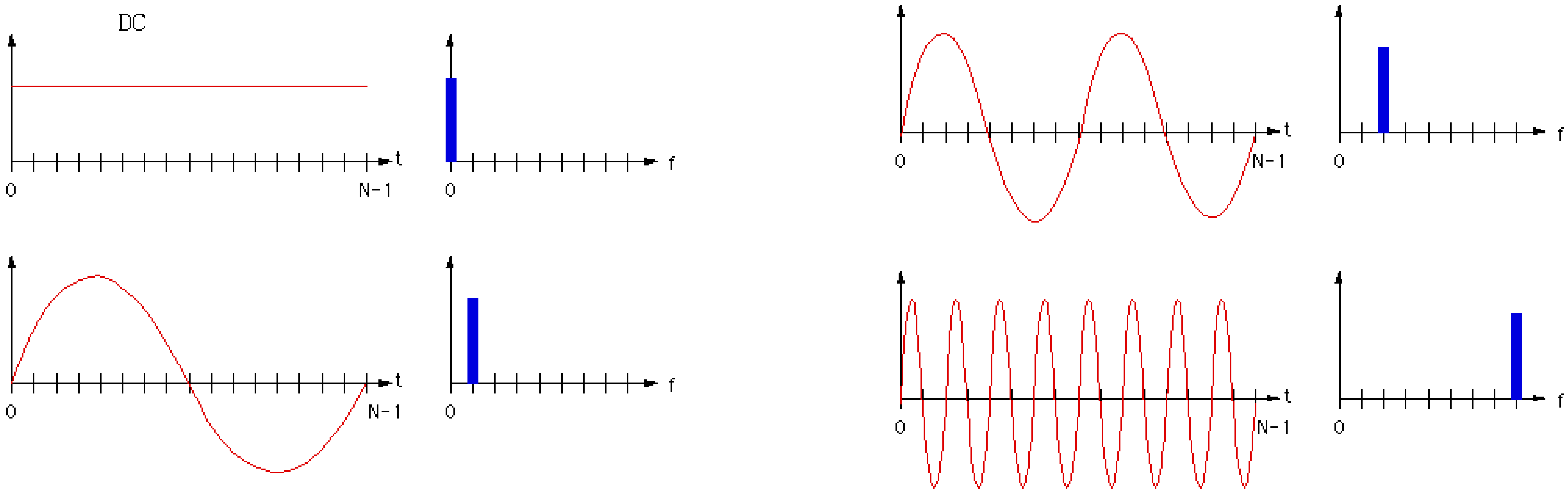
• We mostly skip the middle steps



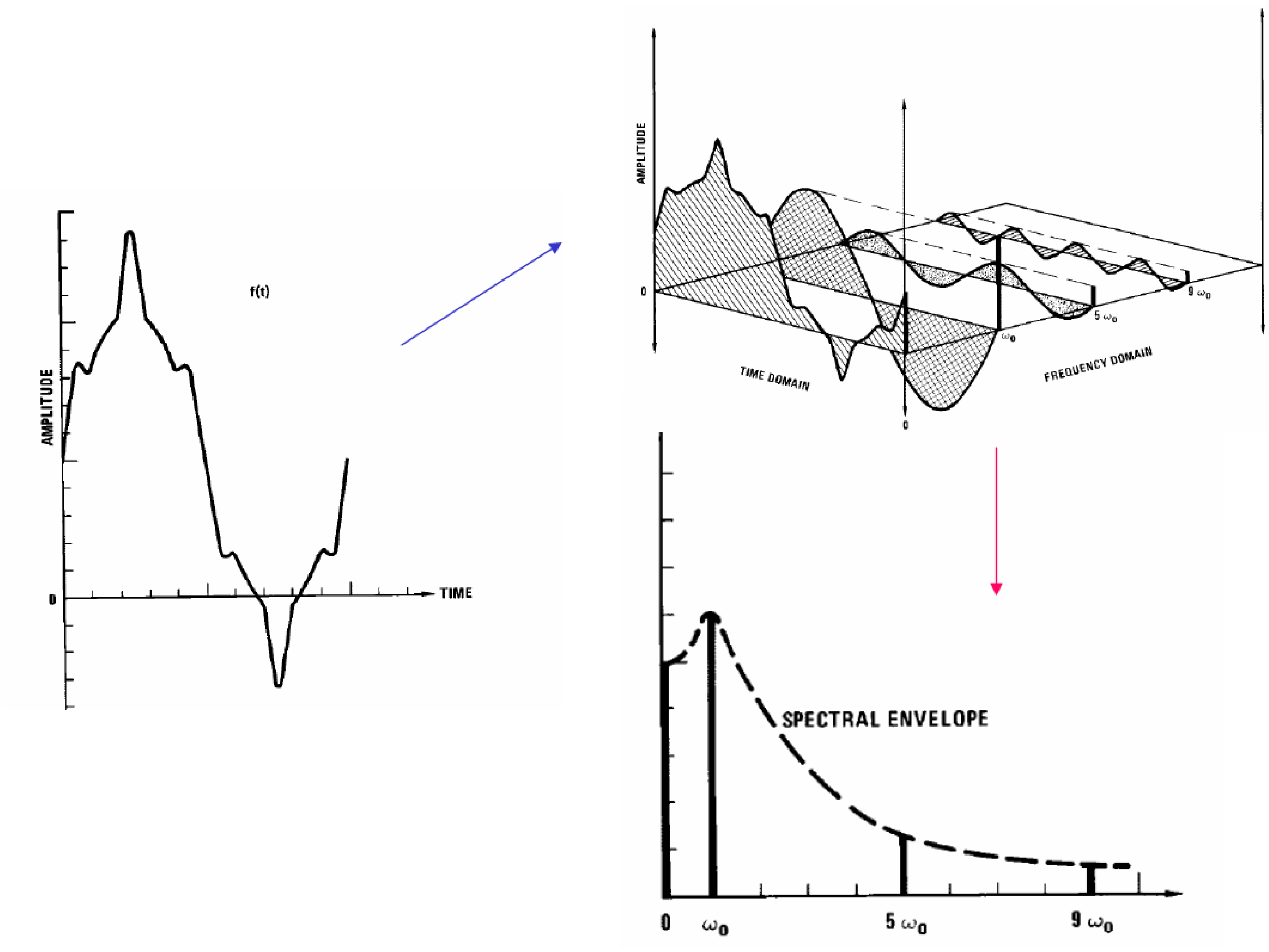


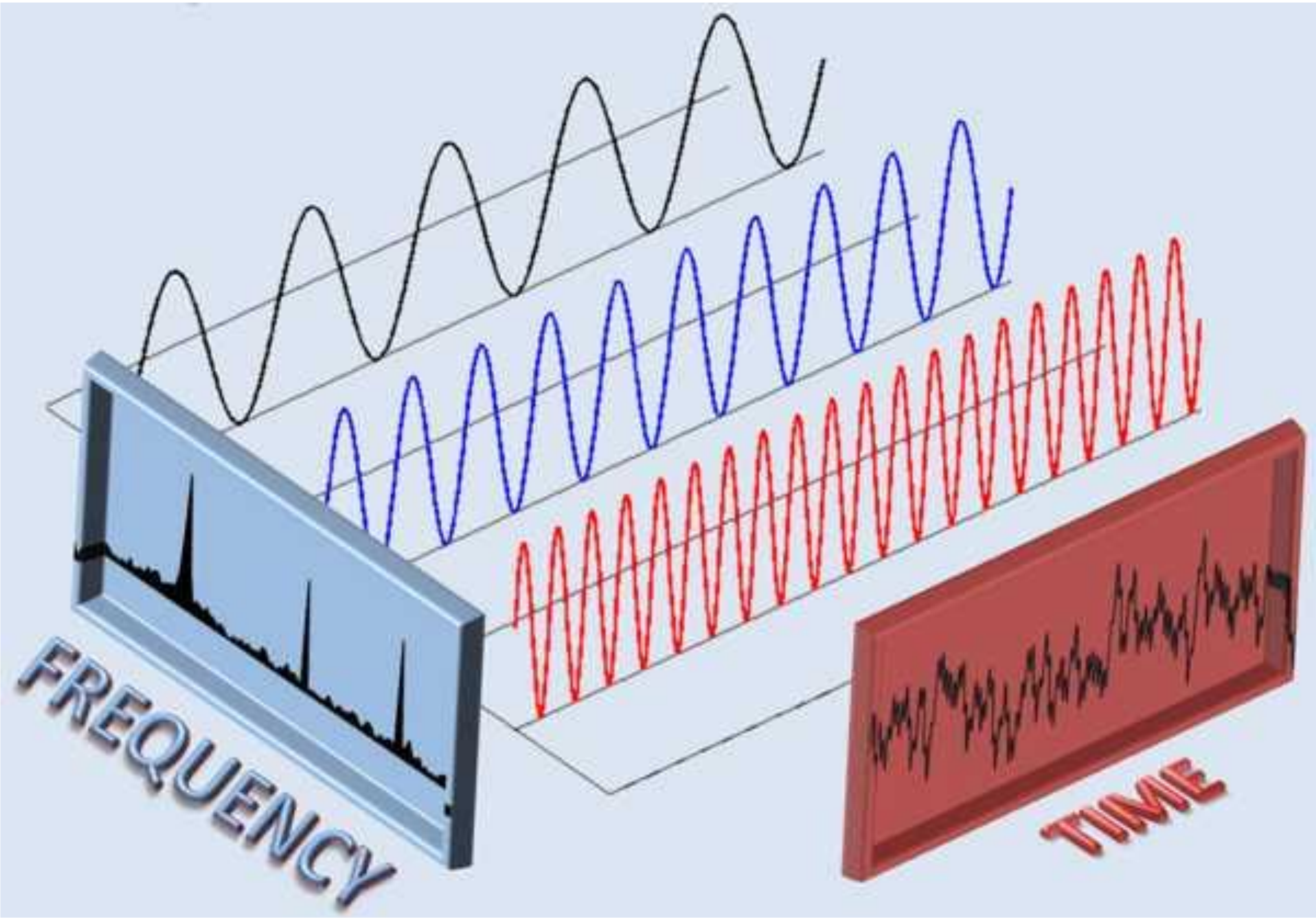
Fourier Transform = ?

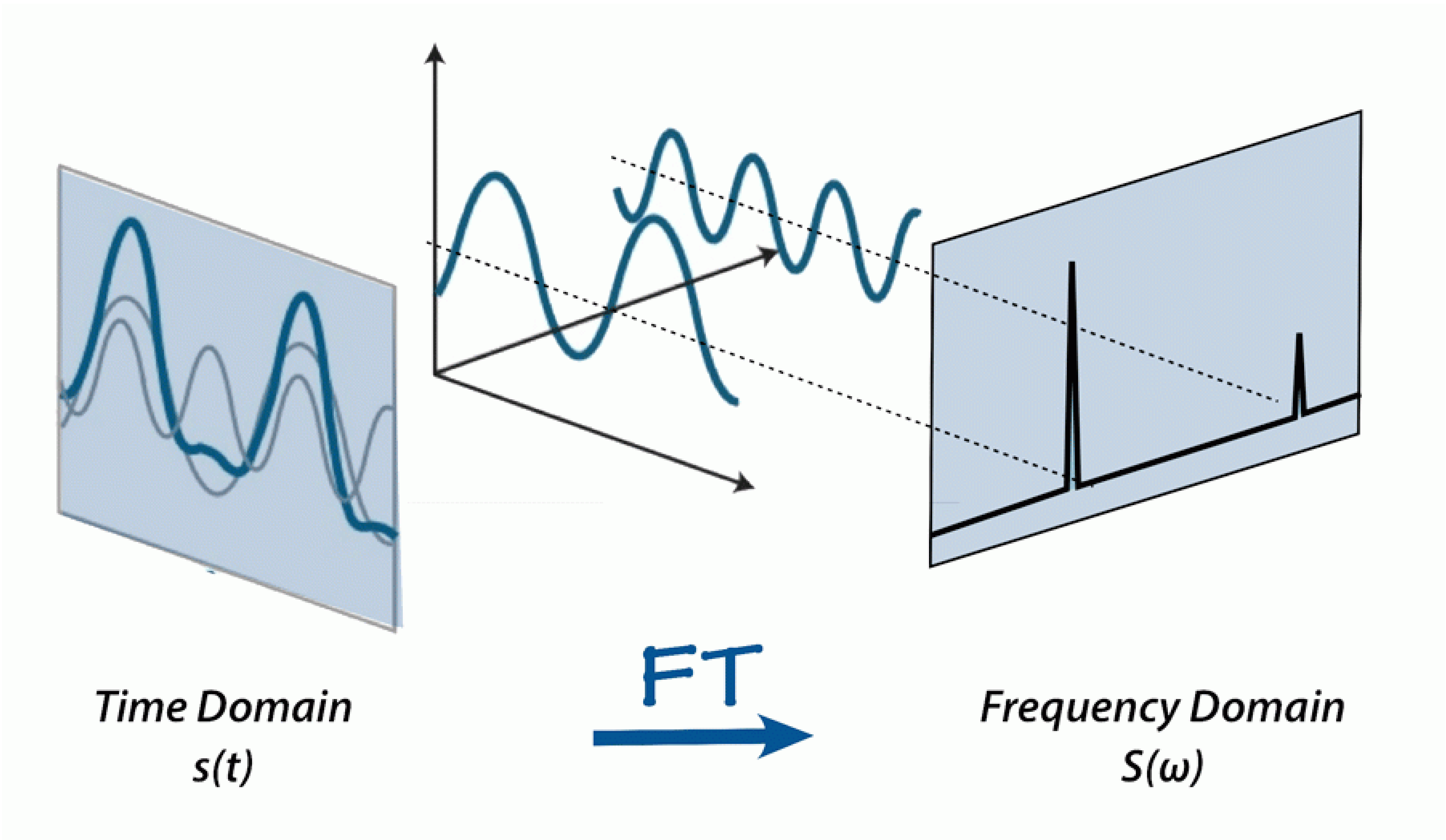
Some Fourier Transforms (Visual)

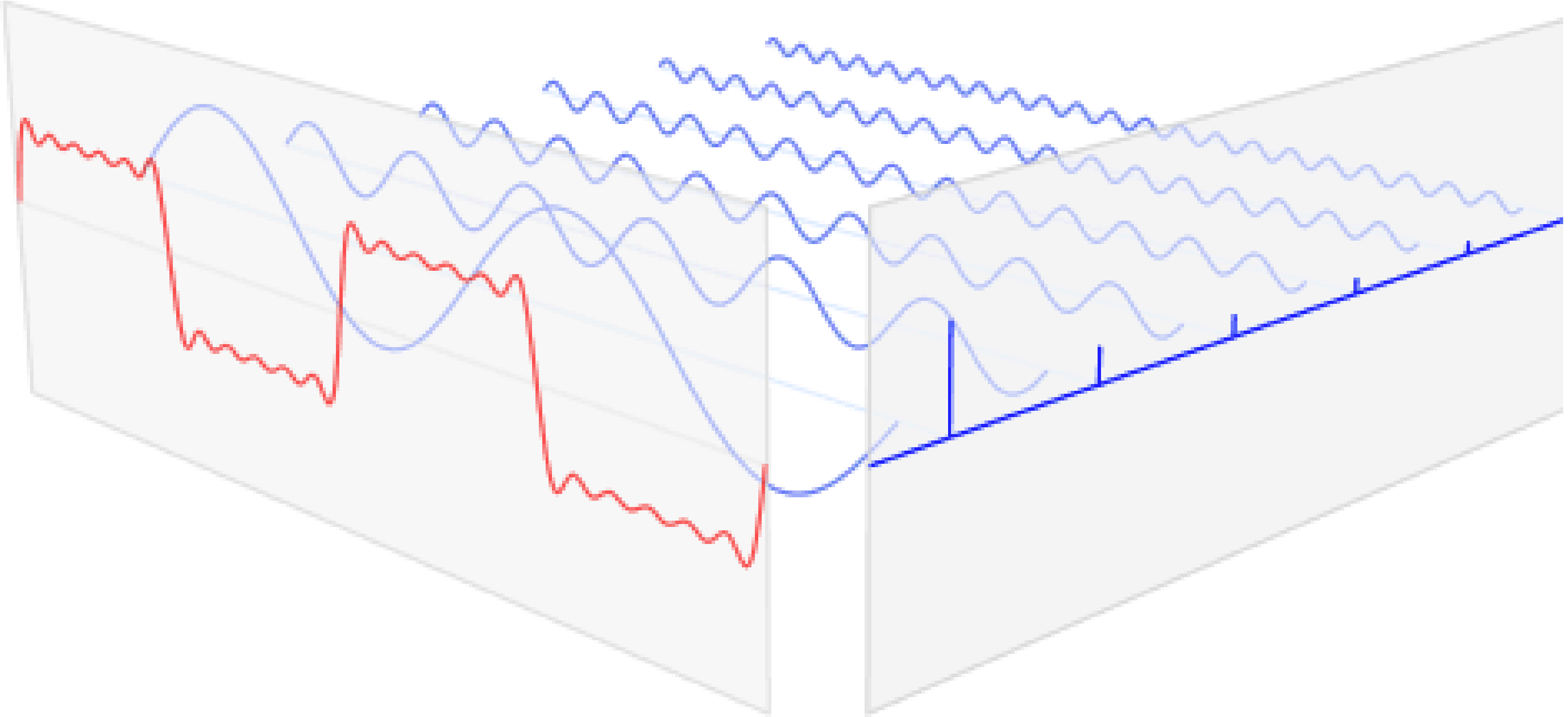


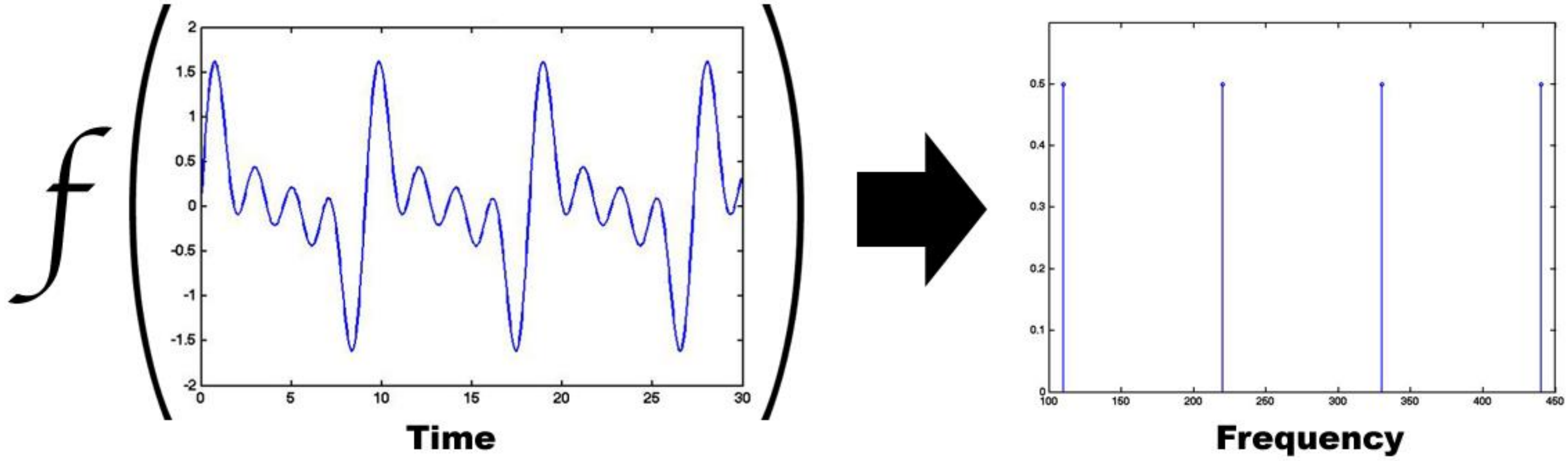
As frequencies increase, the FT peaks move outwards


AMPLITUDE
f(t)
0
TIME
AMPLITUDE
0
TIME DOMAIN
FREQUENCY DOMAIN
SPECTRAL ENVELOPE
0
ω_0
$5\ \omega_0$
$9\ \omega_0$

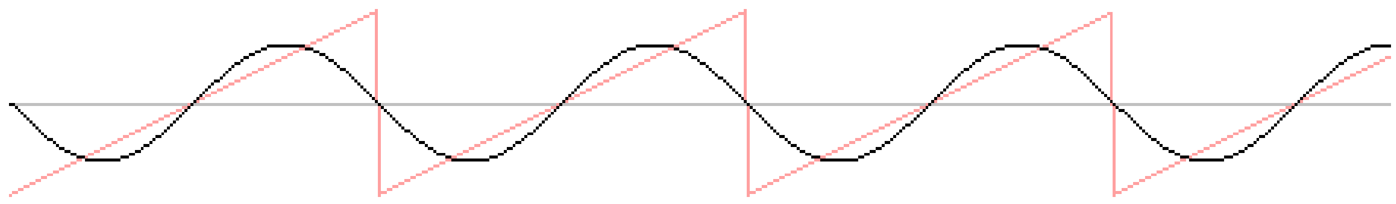

FREQUENCY
TIME

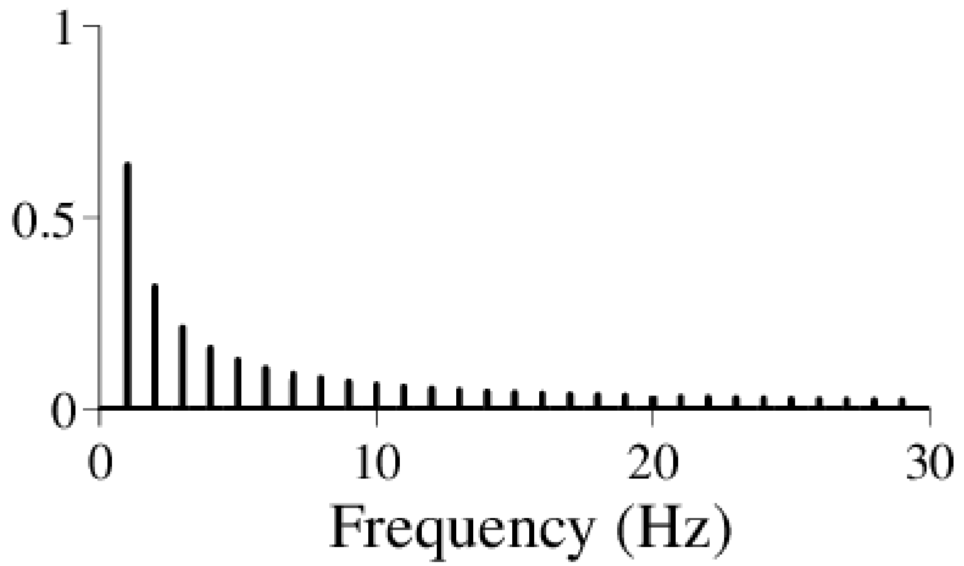

FT
Time Domain
s(t)
Frequency Domain
S(ω)

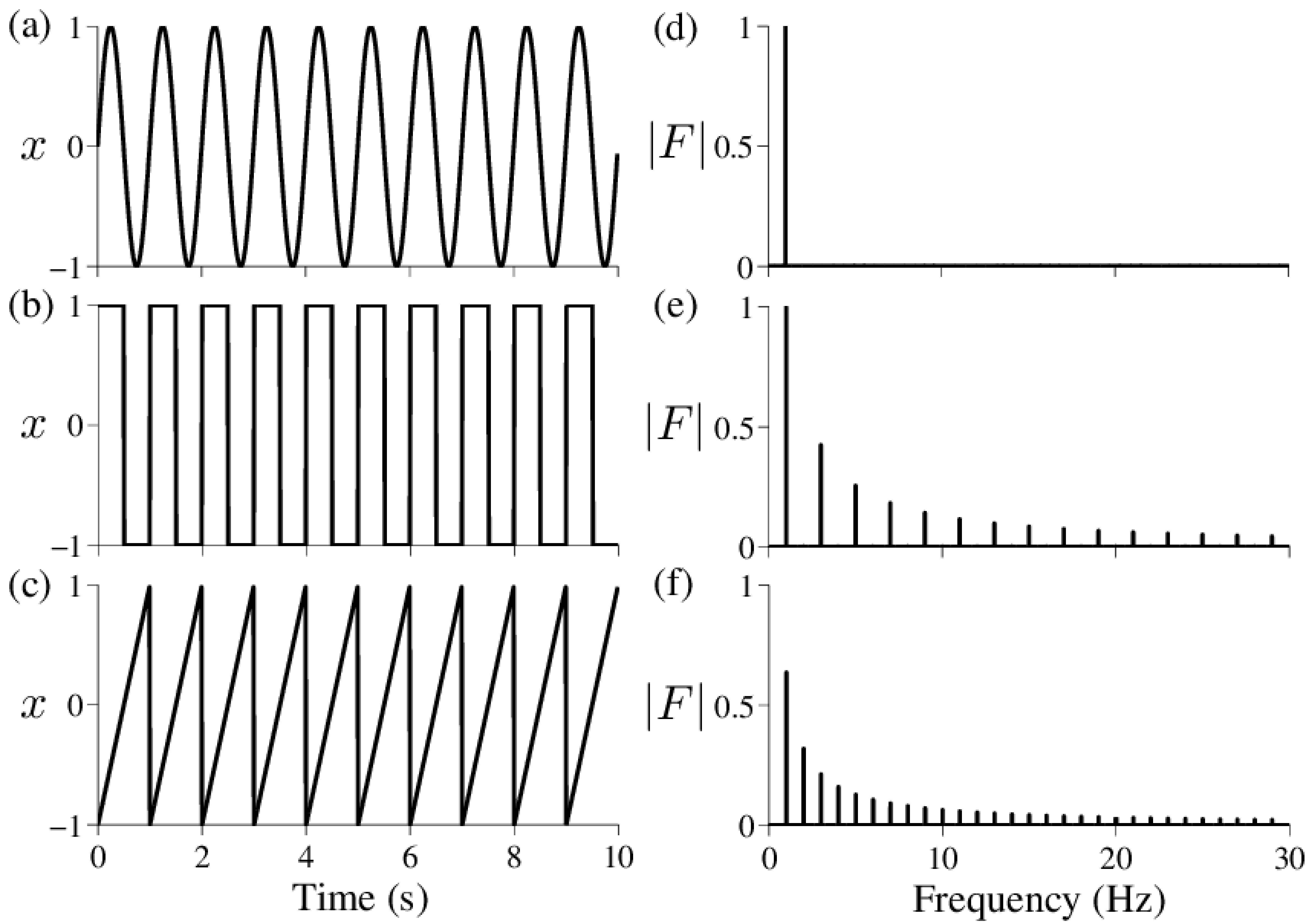


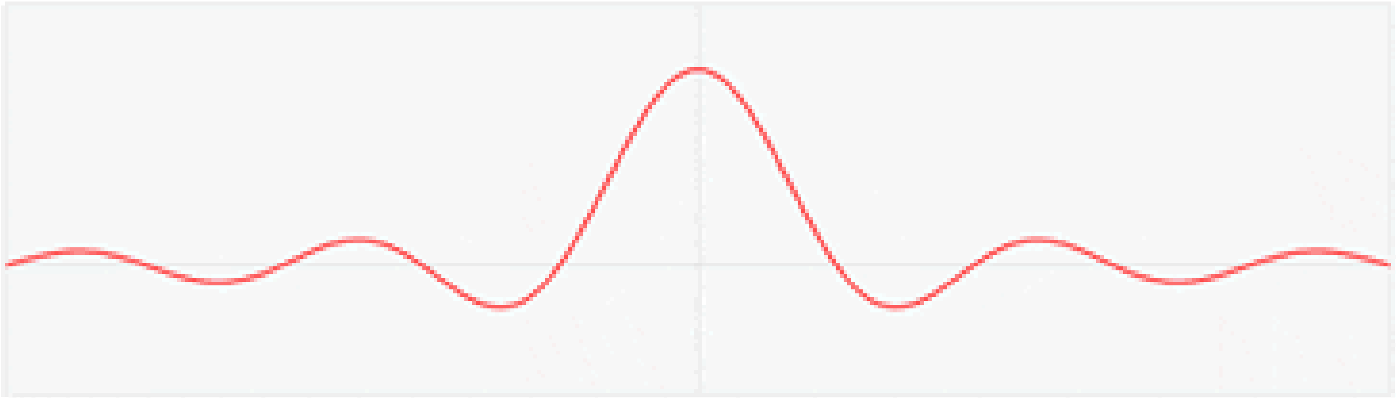

f
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15
20
25
30
Time
0.5
0.4
0.3
0.2
0.1
0
100
150
200
250
300
350
400
450
Frequency

harmonics: 1






(a)
(b)
(c)
(d)
(e)
(f)
x
$|F|$
1
0
−1
0.5
0
2
4
6
8
10
Time (s)
20
30
Frequency (Hz)



$$f(x)$$

1ucasvb.tumblr.com

Q. Is it always possible to write signals as sums of sinusoids?

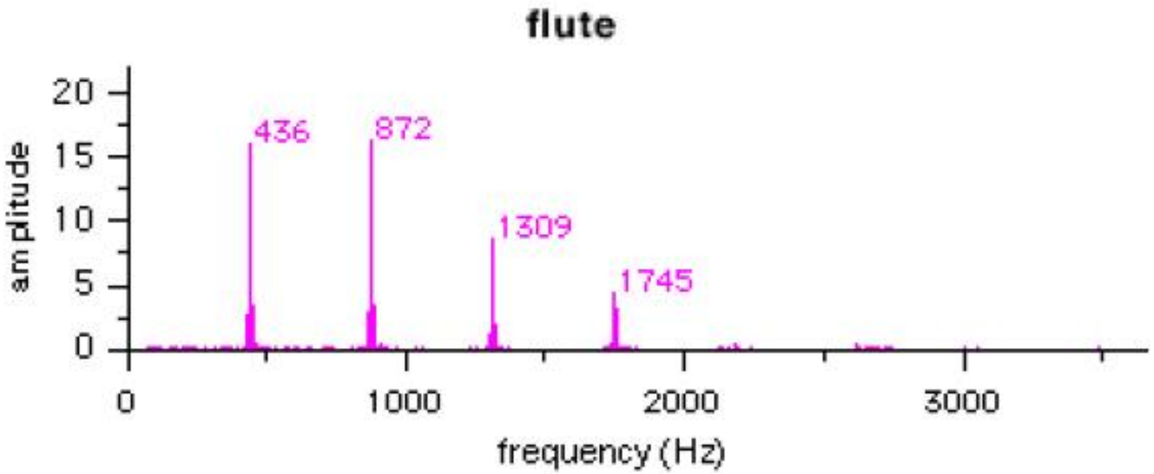
Q. Is it always possible to write signals as sums of sinusoids?

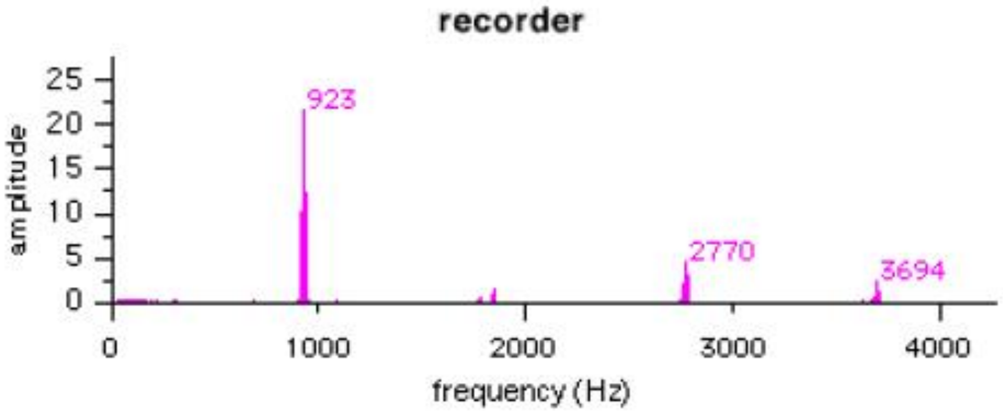
- **No**. There are theoretical signals that do not have a Fourier Transform (e.g., $e^{-at}u(t)$ with $a < 0$).

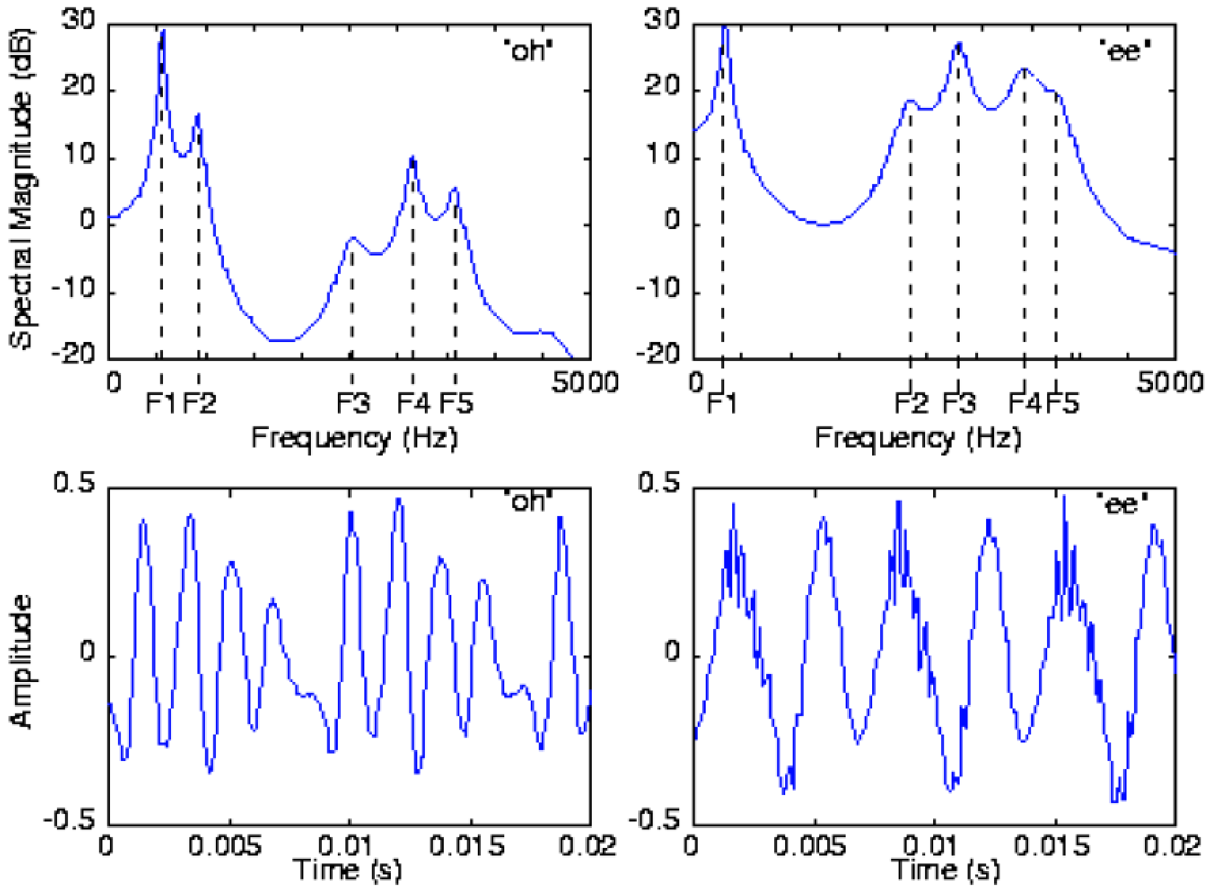
- **However**, all physically realizable signals have Fourier Transforms.

Q. Why Use Periodic Functions (frequencies)?

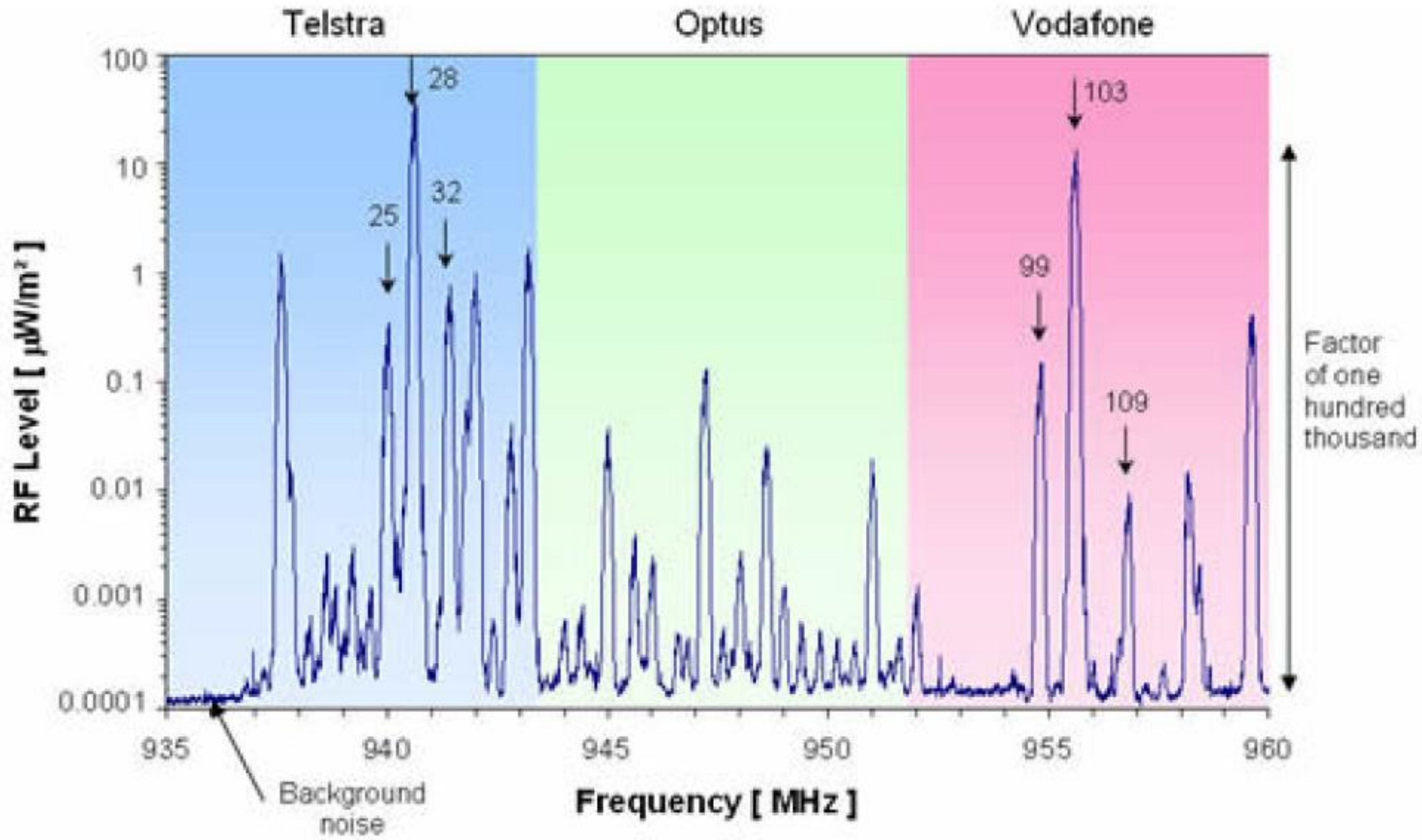
A large number of physical phenomena have underlying periodicities (frequencies)…


flute
amplitude
20
15
10
5
0
436
872
1309
1745
0
1000
2000
3000
frequency (Hz)

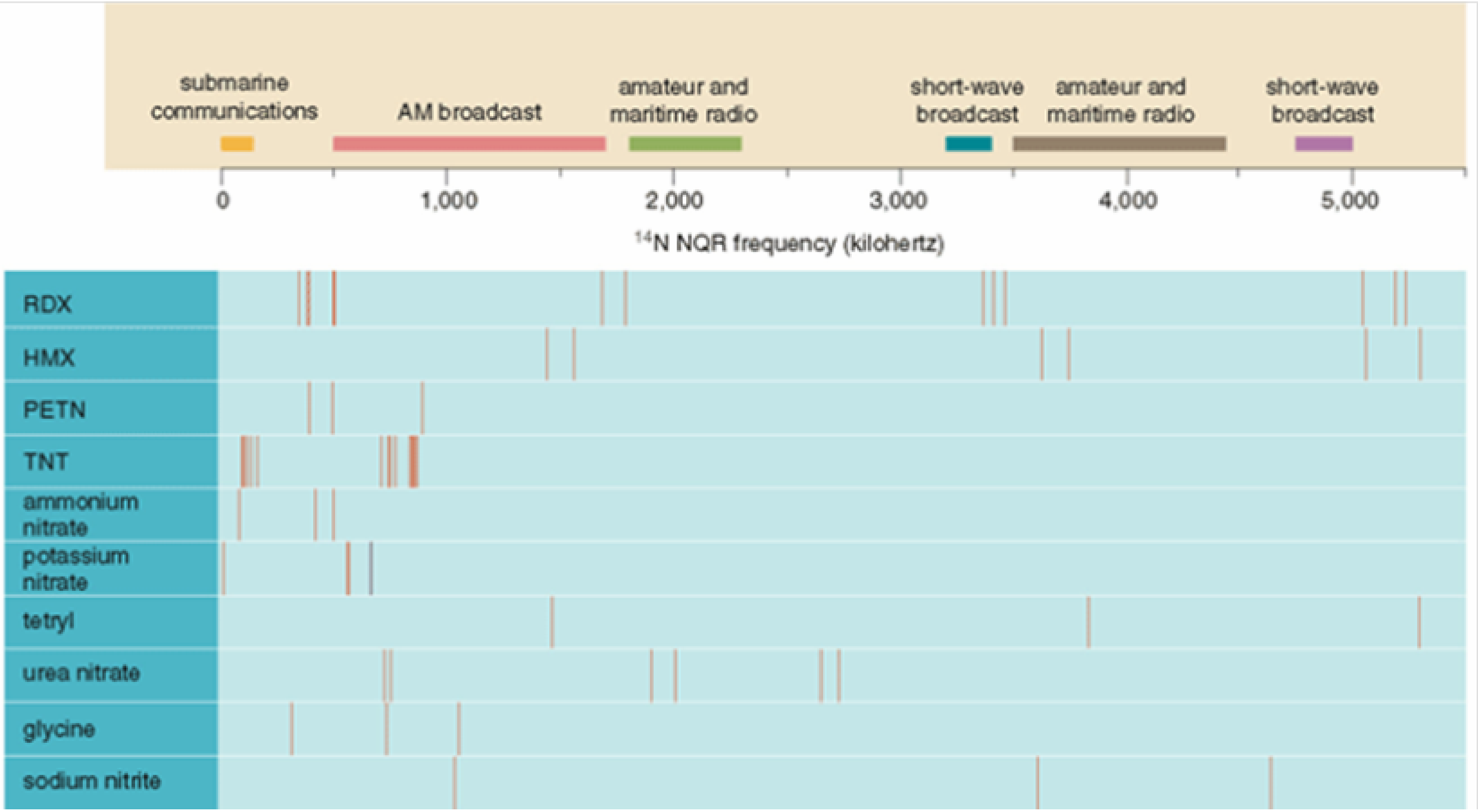

recorder
amplitude
25
20
15
10
5
0
923
2770
3694
0
1000
2000
3000
4000
frequency (Hz)



Voice Recognition

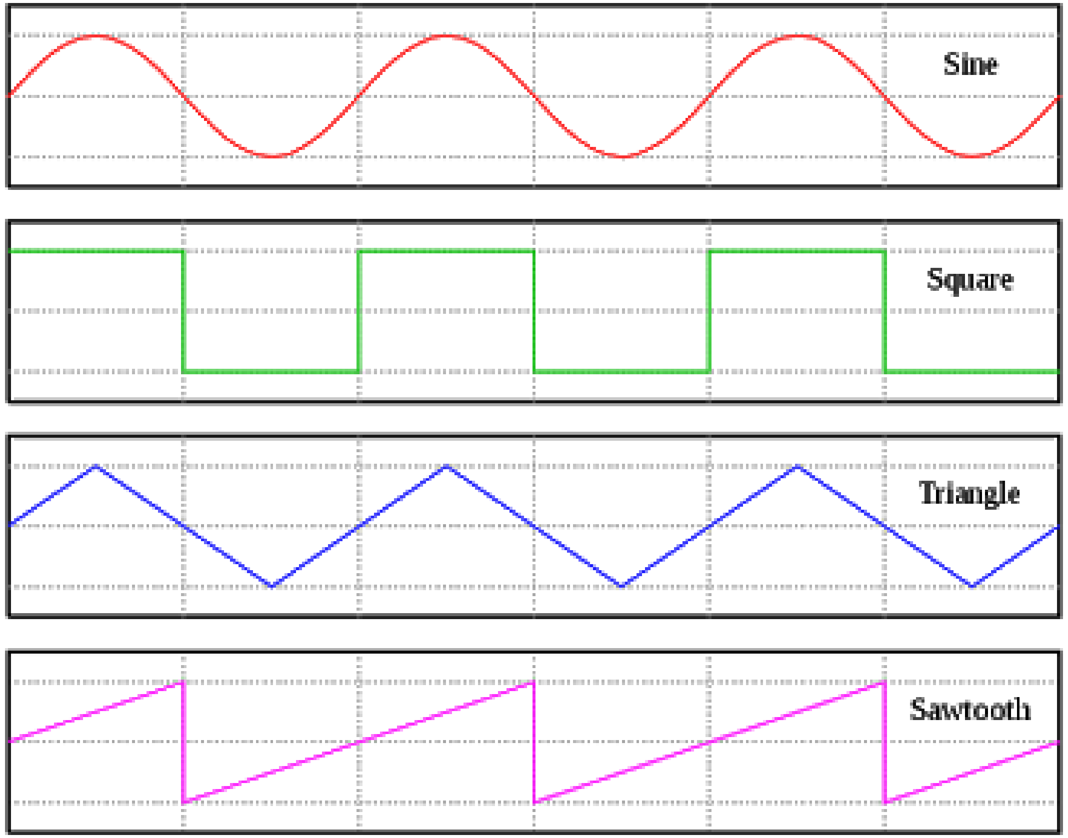


Communication systems are based on the frequencies of tunable antennas



Chemicals may be identified by the unique resonant frequencies of their nuclei or molecules

Q. Why Sinusoids?

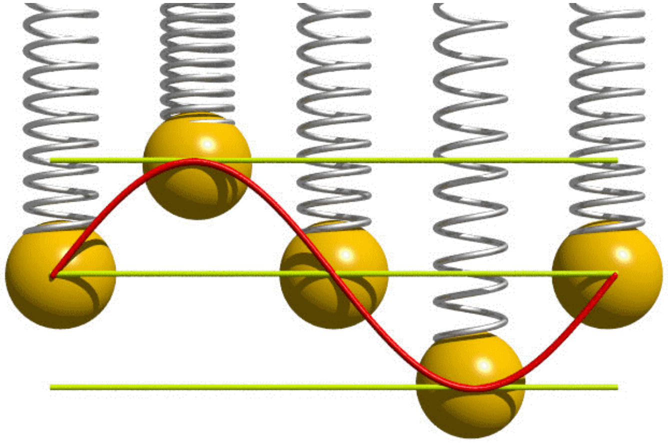


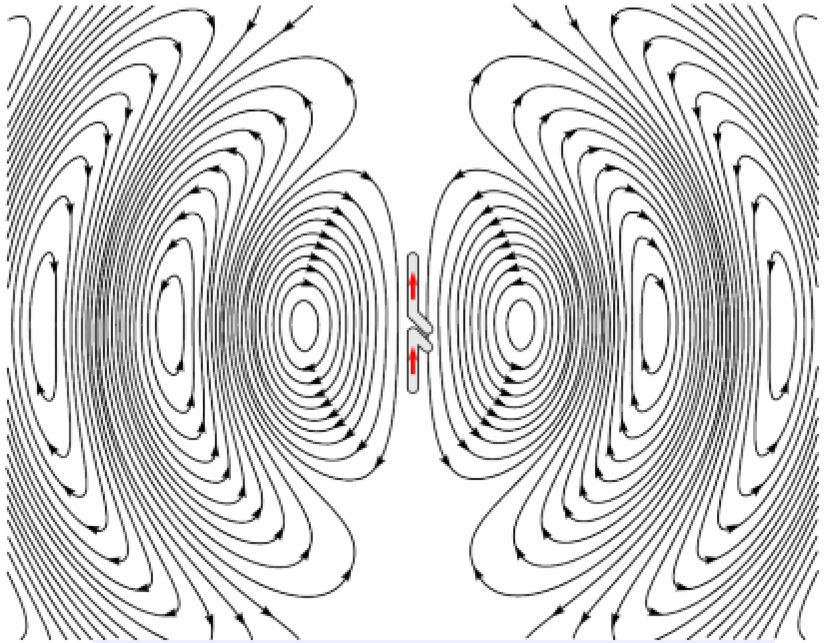
Why not other types of periodic functions?

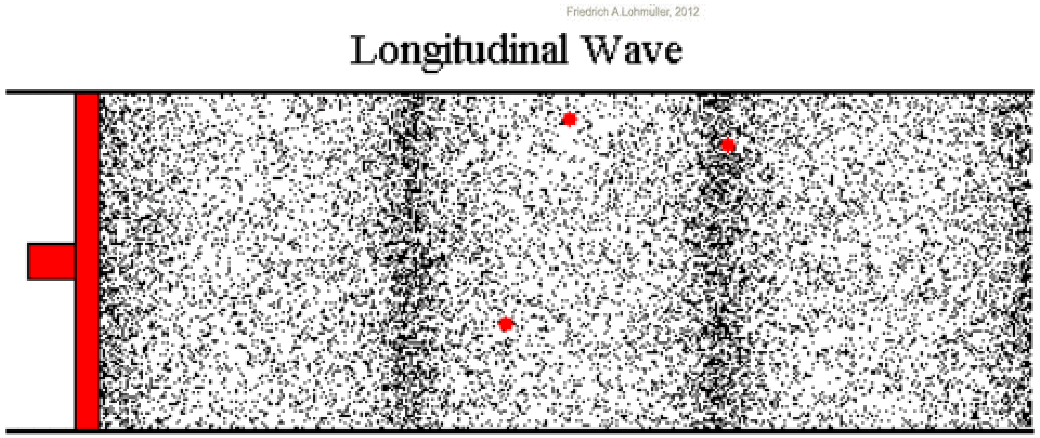
Q. Why Sinusoids?

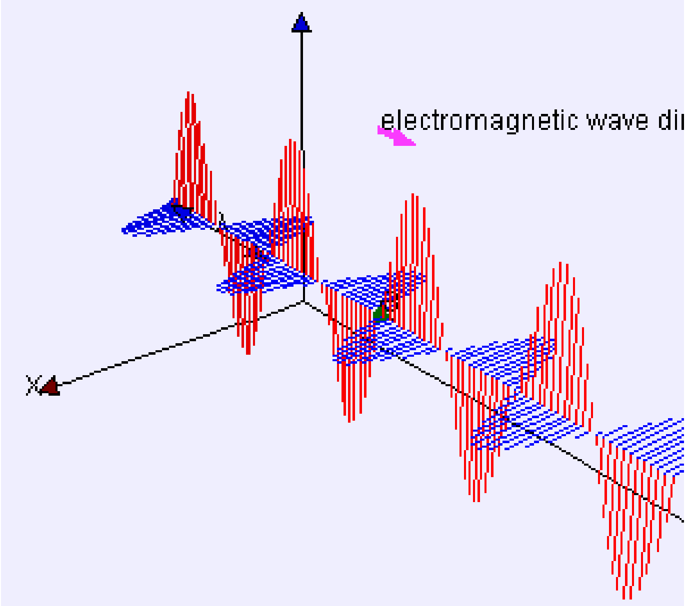
- **Smooth** (analytically simpler, e.g., differentiable, integrable…)
- Nicely reflect behavior of **natural phenomena** (to-and-fro motions)

Q. Why Sinusoids?









In fact, Fourier Transform does not use just sinusoids, it uses ***complex sinusoids****!!!*

Why?

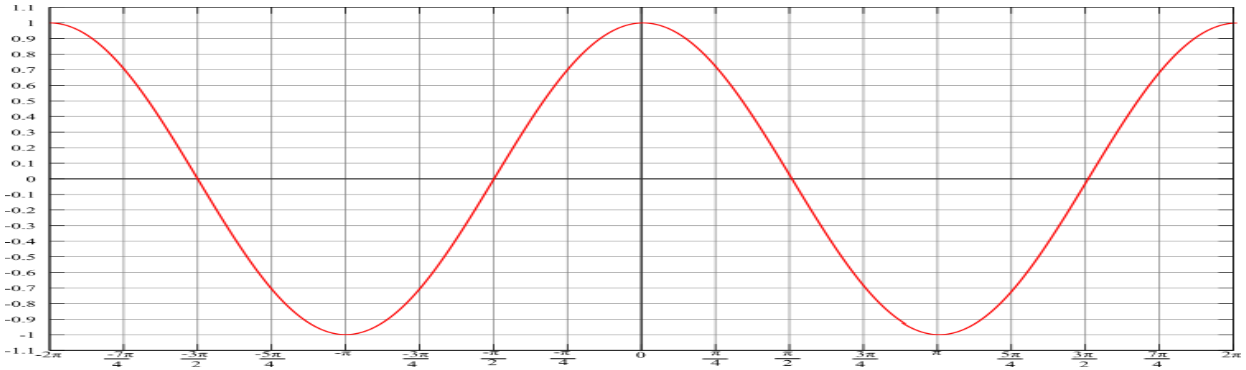
- More general than real sinusoids
- More elegant analytically and in calculations

And what exactly was a complex sinusoid?

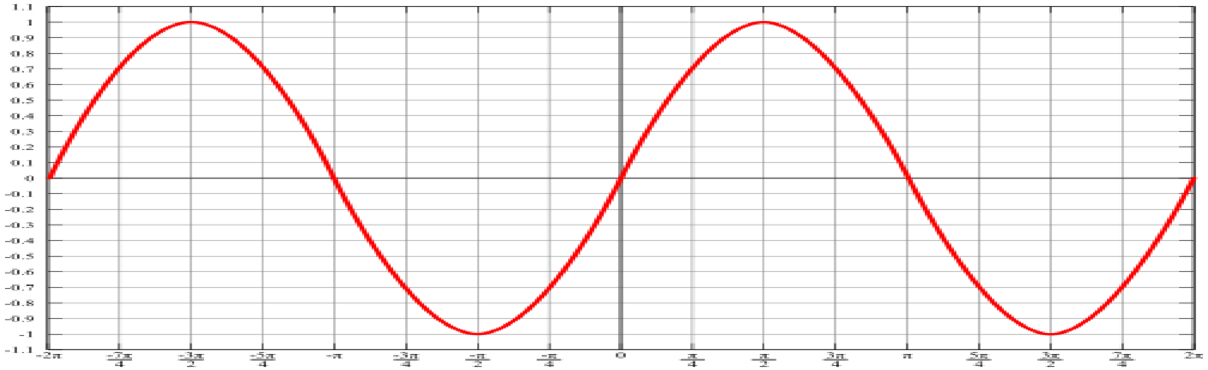
A Sine and a Cosine Walk Into an Imaginary Bar…

The Complex Sinusoid - $e^{j\omega t}$

$\cos(\omega t)$

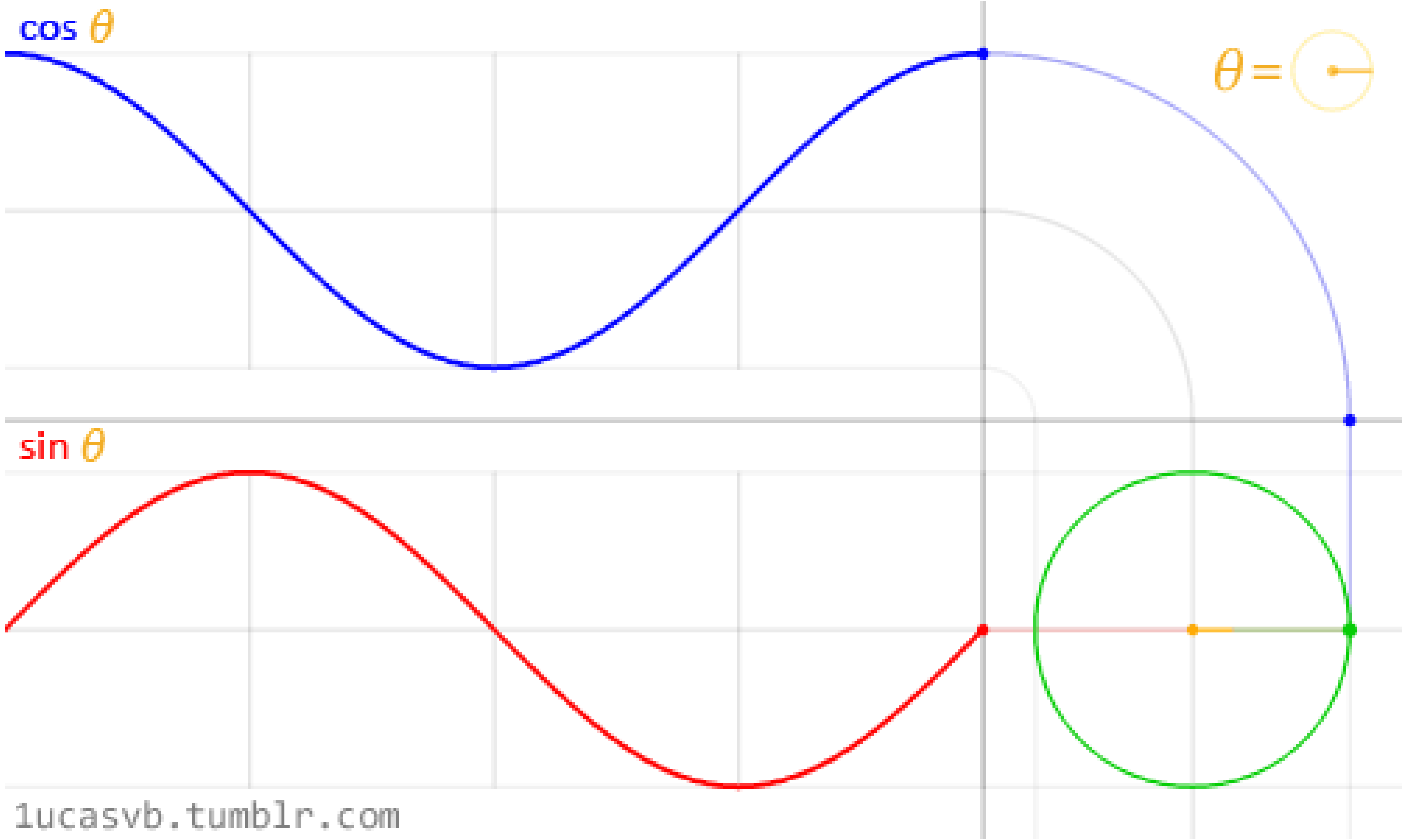


$\sin(\omega t)$

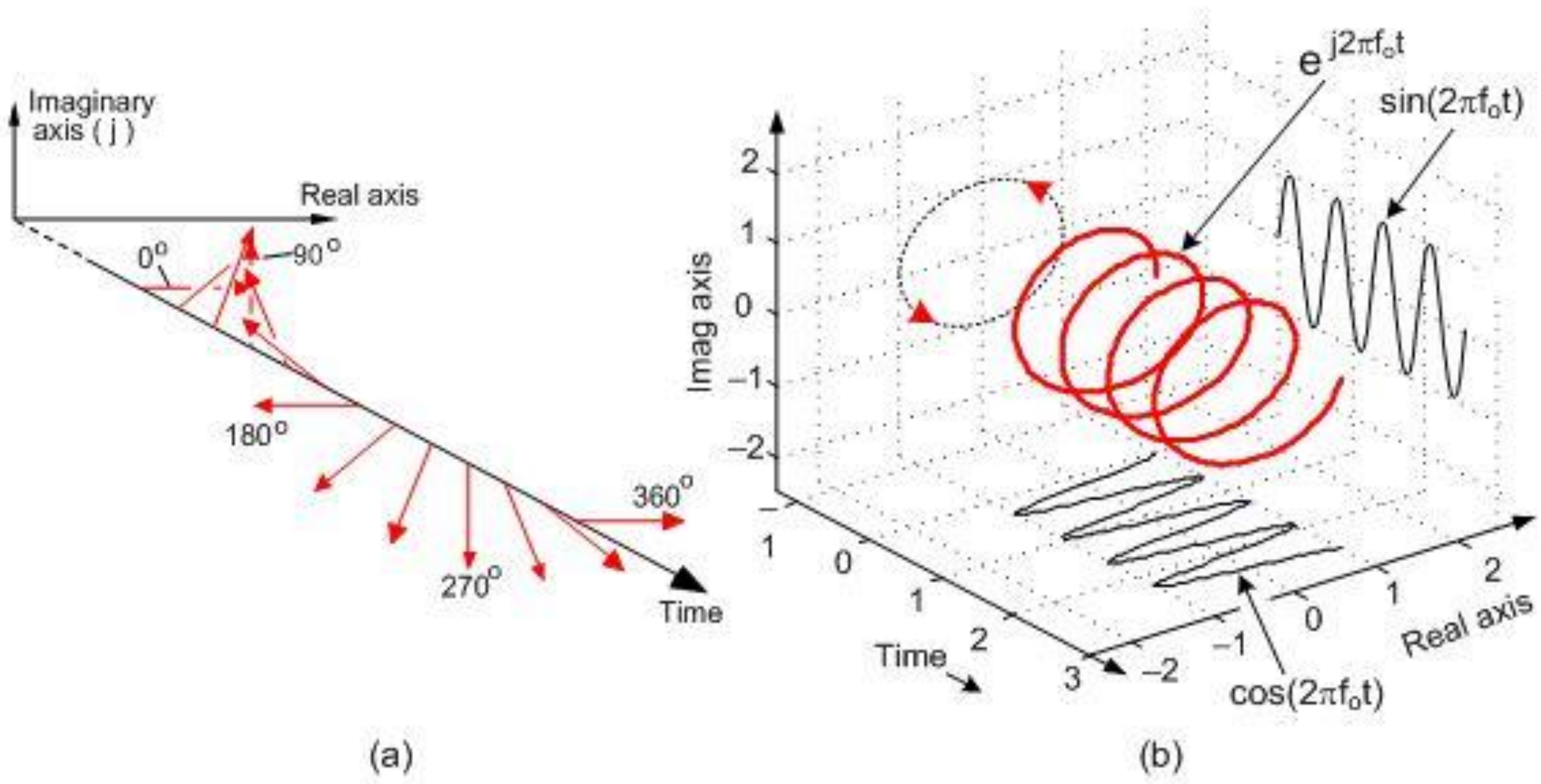


The Complex Sinusoid - $e^{j\omega t}$

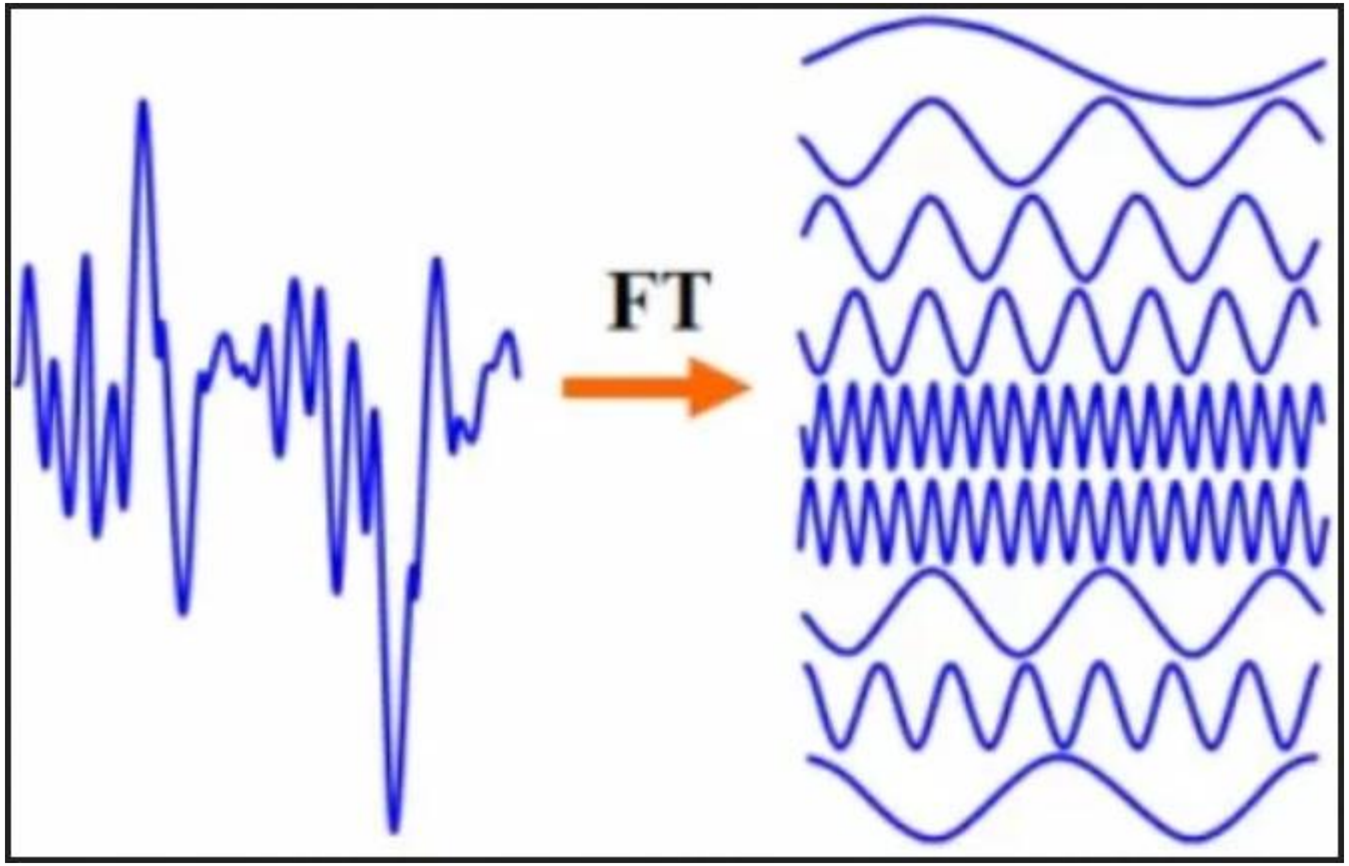
$\cos(\omega t) + j\sin(\omega t)$



The Complex Sinusoid - $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

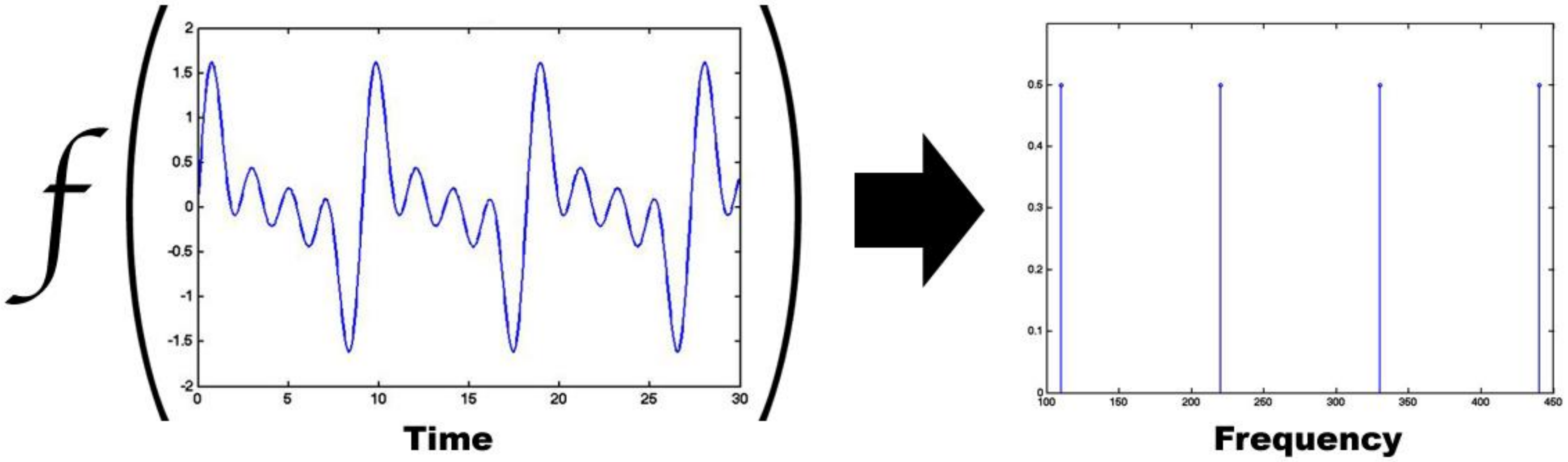


Mathematically Speaking…

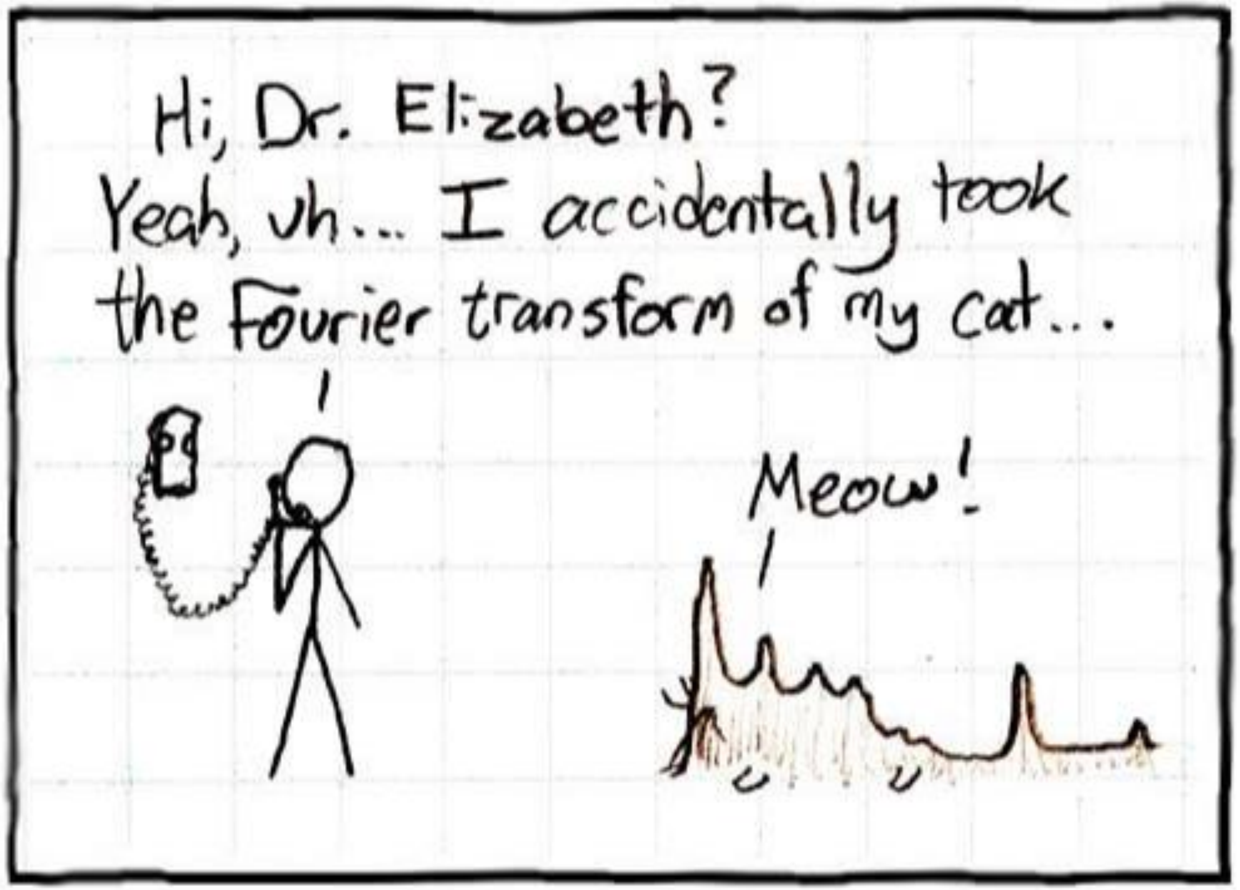


$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Mathematically Speaking…



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$


Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took the Fourier transform of my cat...
Meow!

Questions?? Thoughts??

ES 332
Signals and Systems

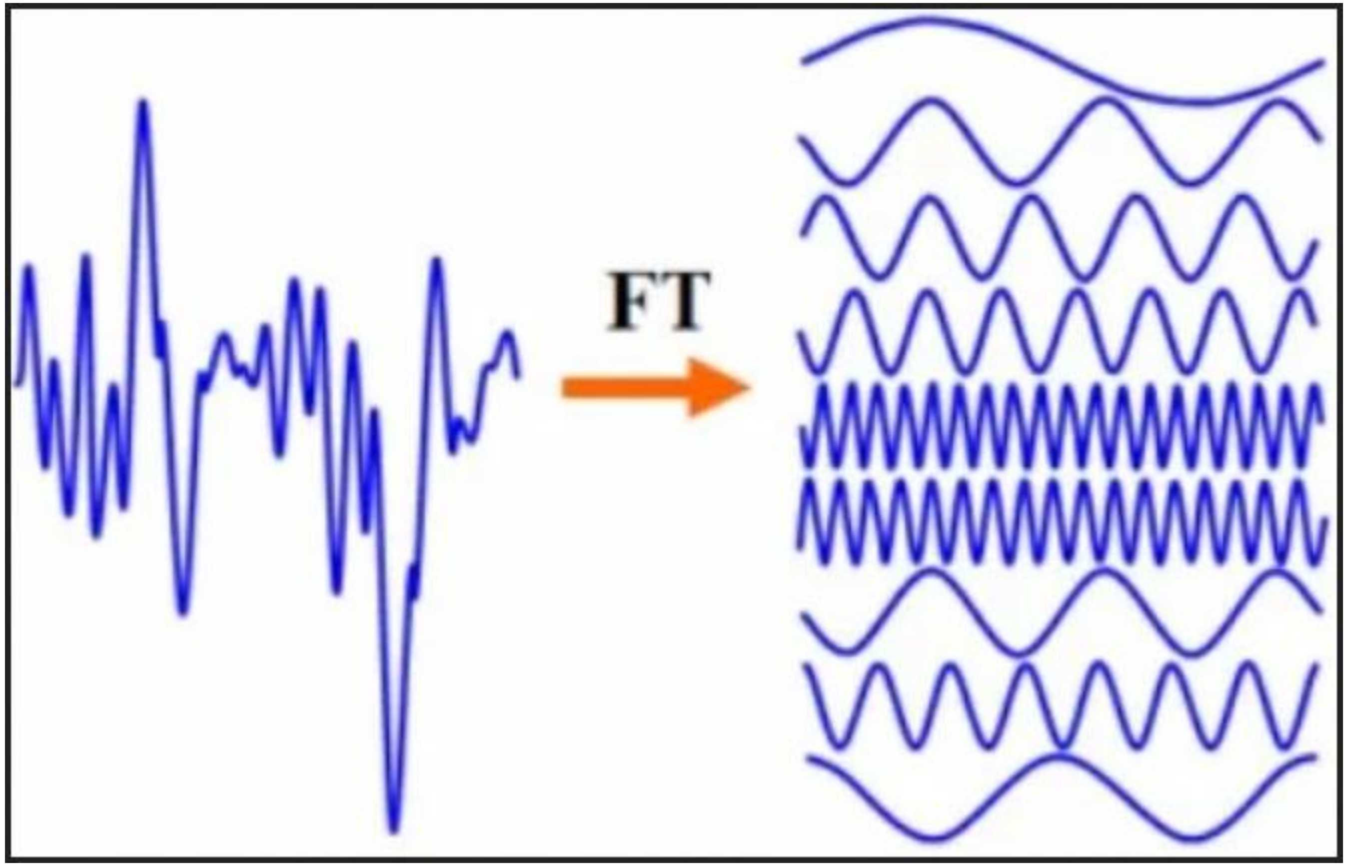
with

Dr. Naveed R. Butt

@

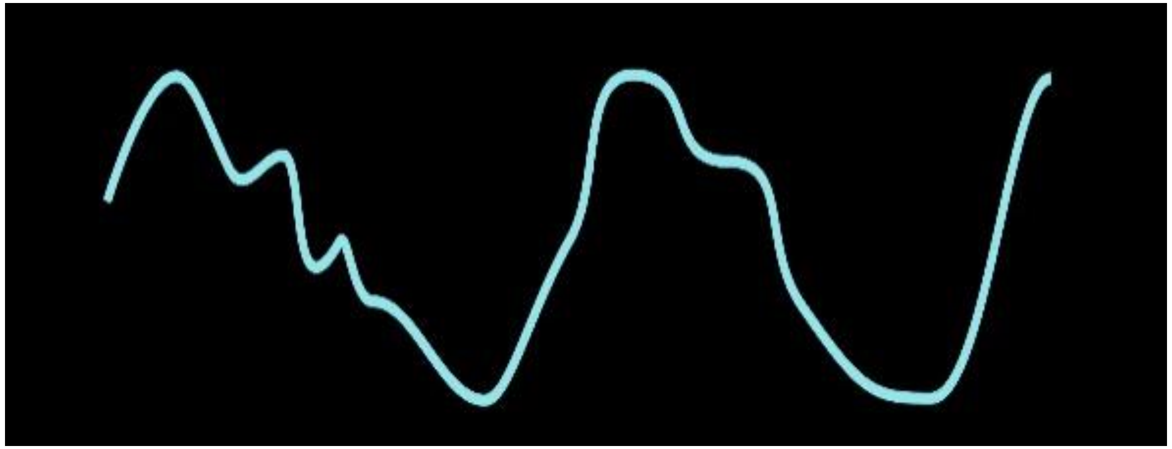
GIKI - FES

Q. Can we write signals as sums of periodic functions (frequencies)?

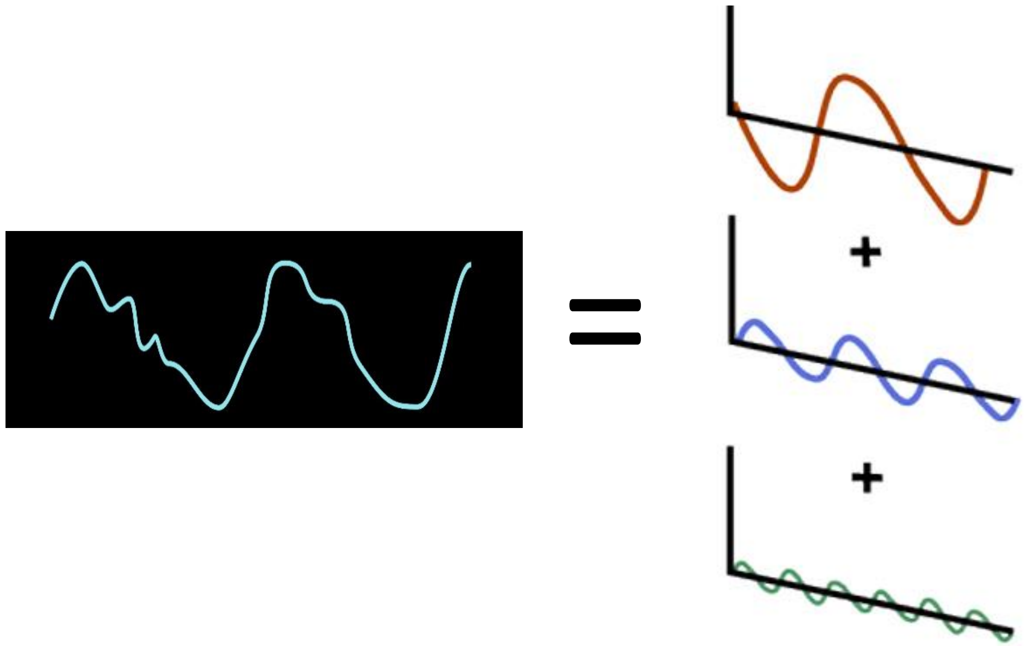


Baking a Fourier Cake

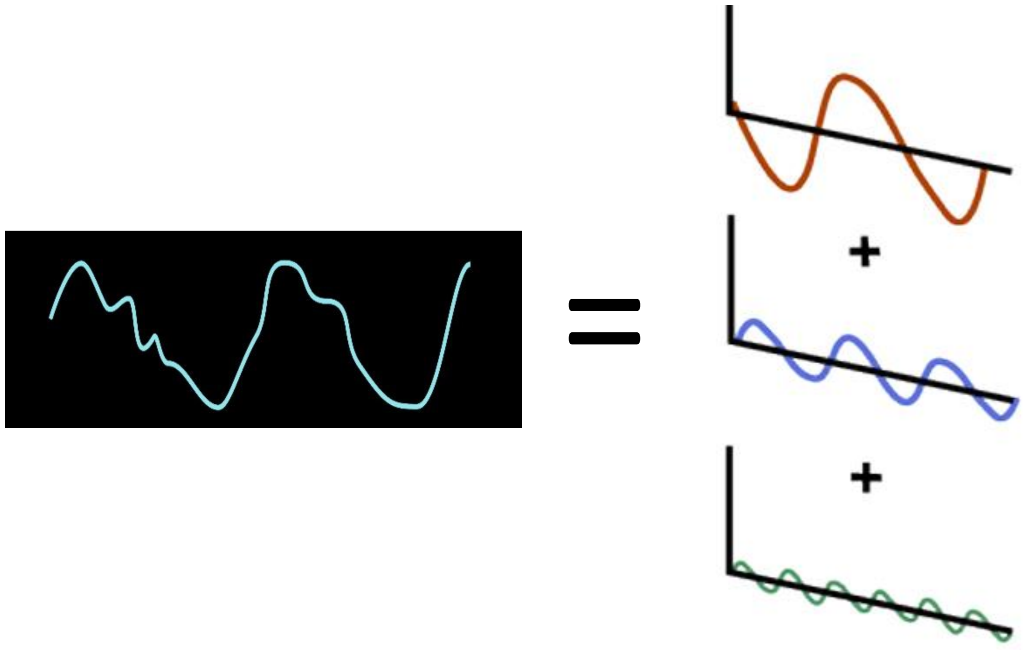
- **Given**: Signal shape (time-domain)
- **Ingredients**: Sinusoids of different frequencies
- **Choose**: How much of the each ingredient (sinusoid) to use?



- In Fourier Transform, we want to look at signals in terms of a fixed set of ingredients
  - Ingredients : Sinusoids of different frequencies

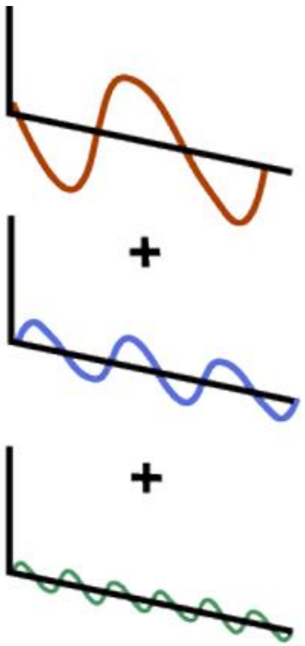


- In Fourier Transform, we want to look at signals in terms of a fixed set of ingredients
  - Ingredients : Sinusoids of different frequencies

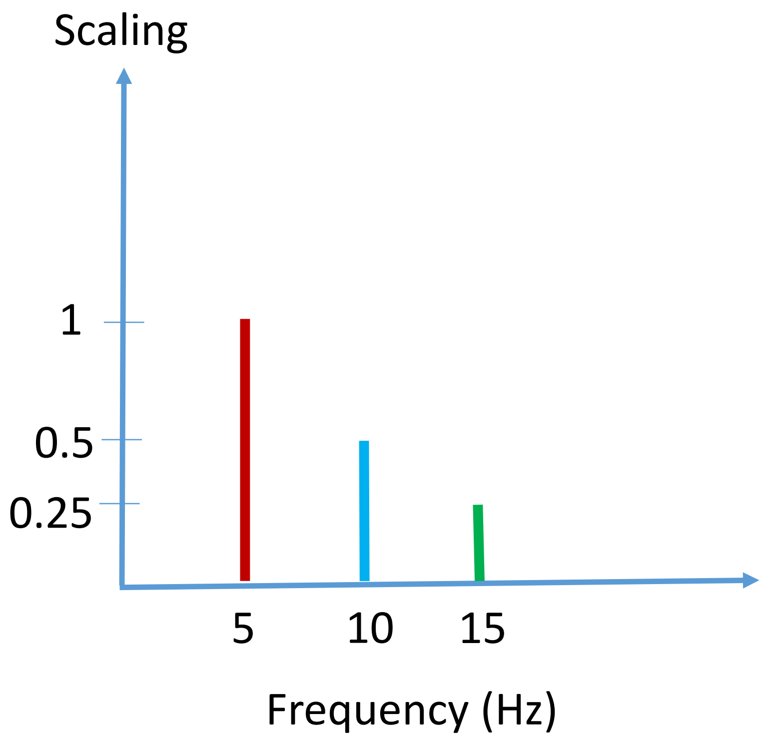


Ingredient (sinusoid frequency)	Amount (scaling)	Process
f_1	1	**Add all**
f_2	0.5	
f_3	0.25	

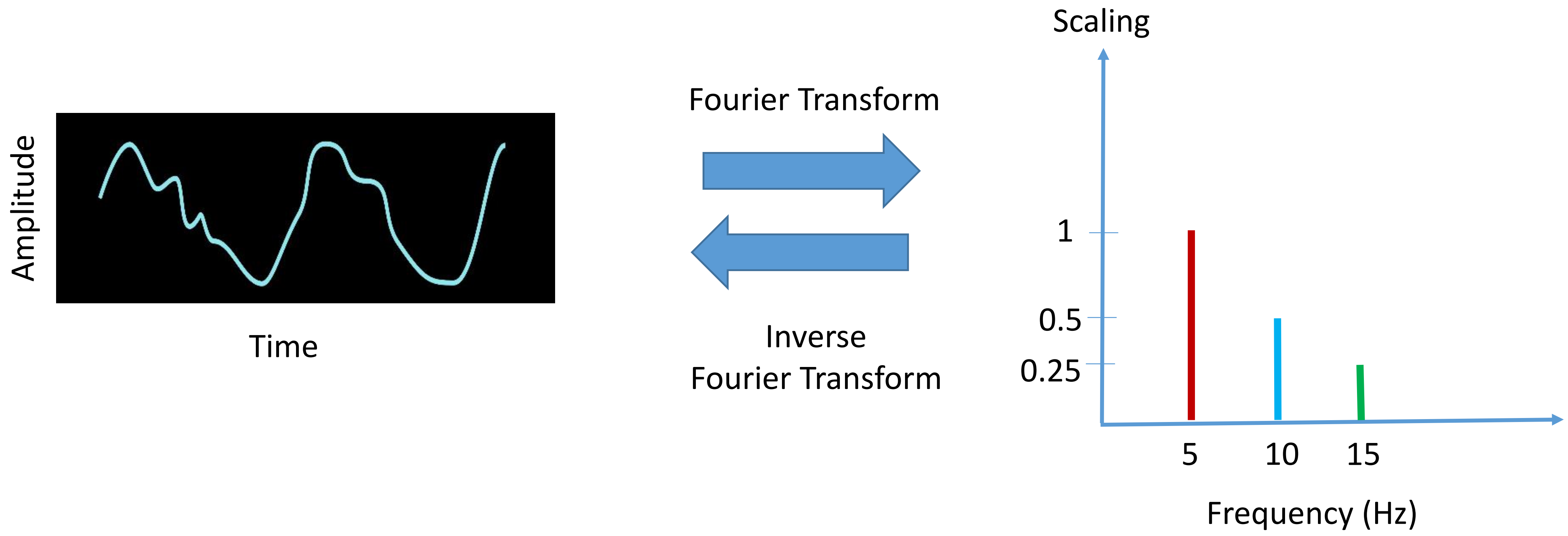
- How is this shown after Fourier transform?



Ingredient (sinusoid frequency)	Amount (scaling)	Process
5 Hz	1	**Add all**
10 Hz	0.5	
15 Hz	0.25	



- We mostly skip the middle steps

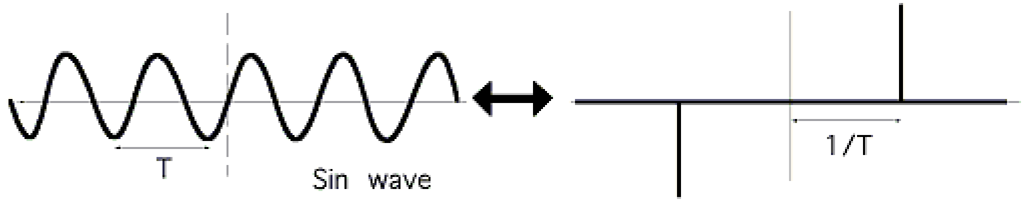


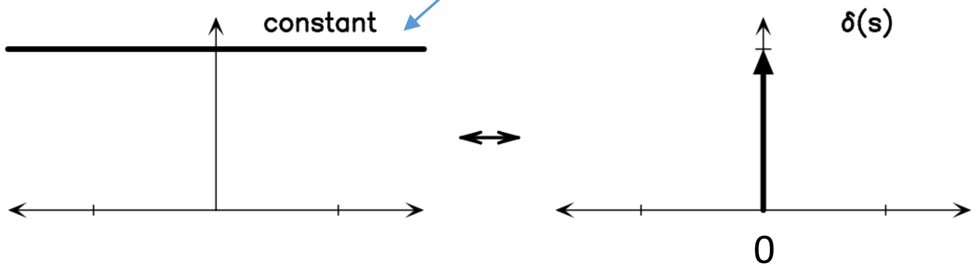
FT – Pairs and Rules-of-Thumb

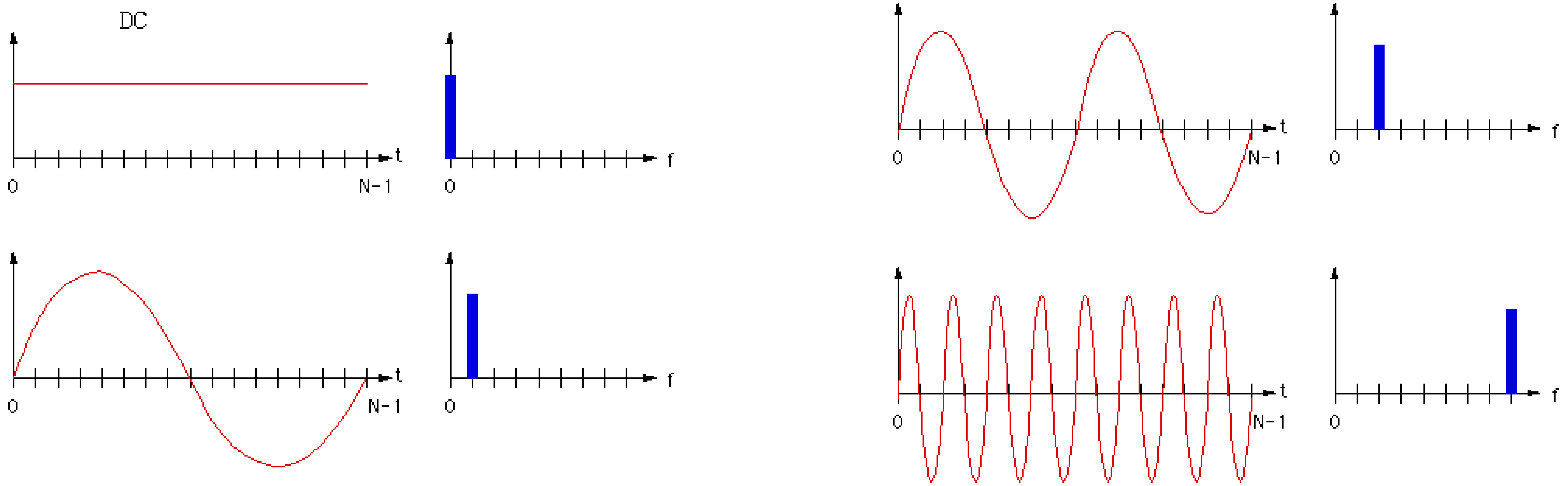
Some Fourier Transforms (Visual)

Not changing
(frequency = 0)

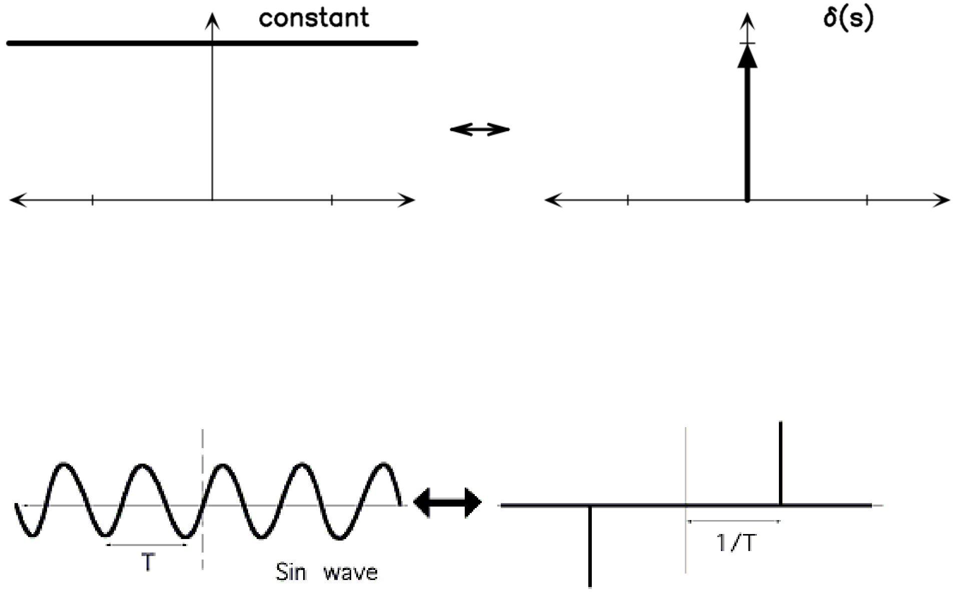
$$\sin(\omega t) \;=\; \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

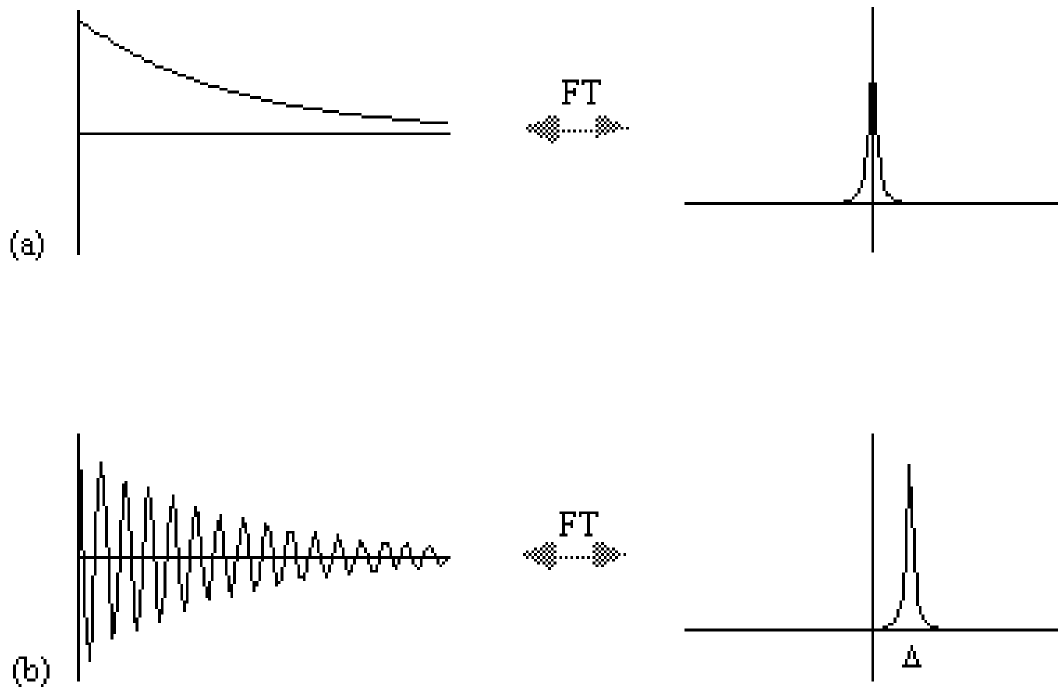




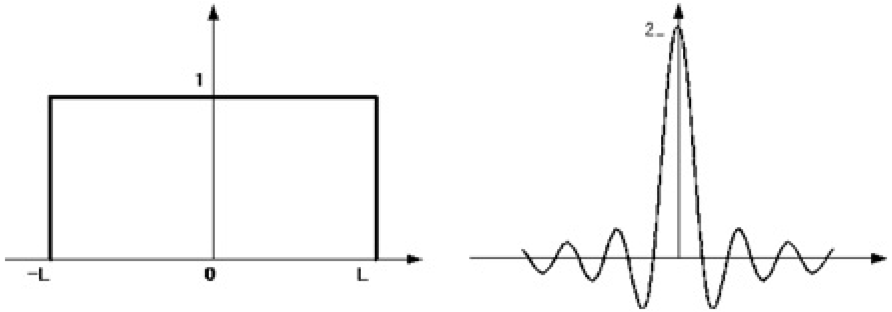


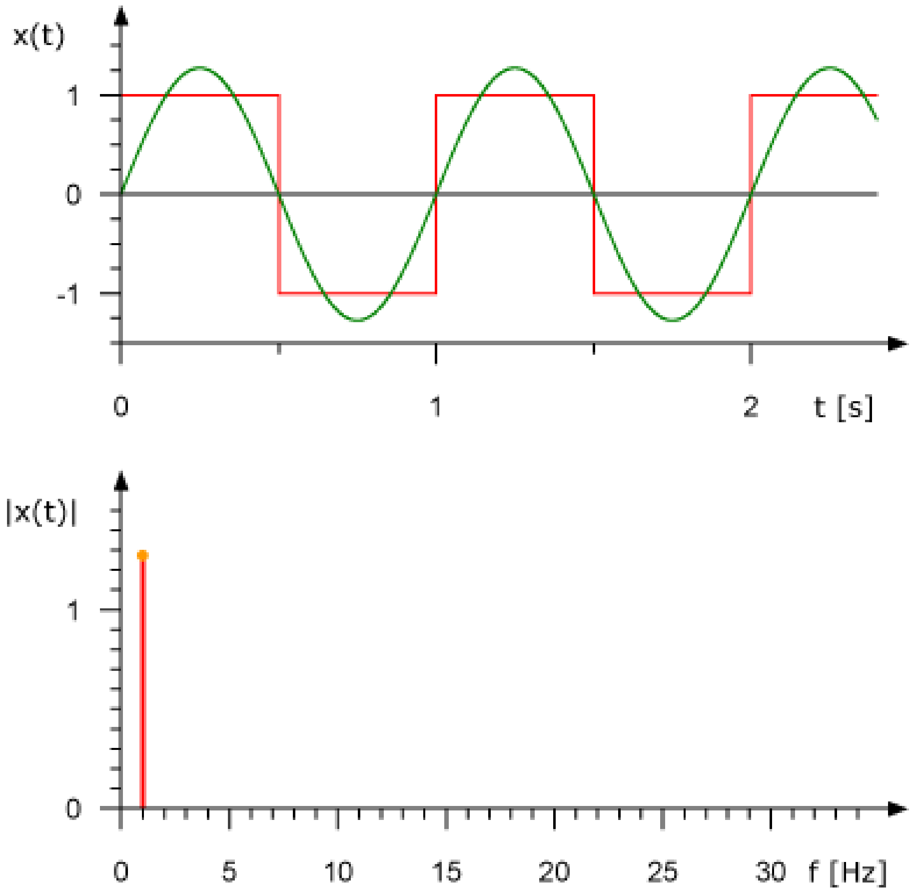
Rule1: *As frequency increases, the FT peaks move outwards*



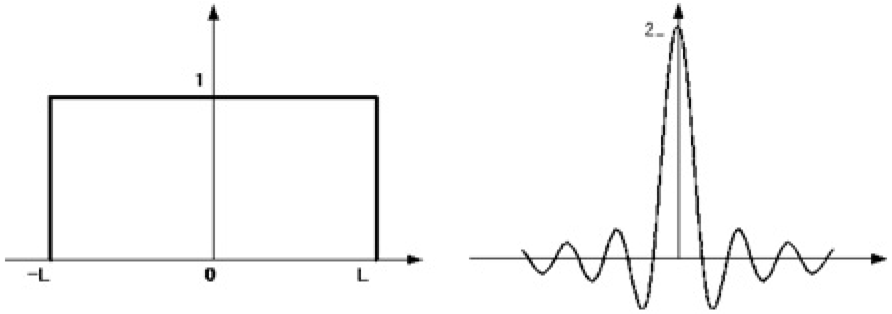


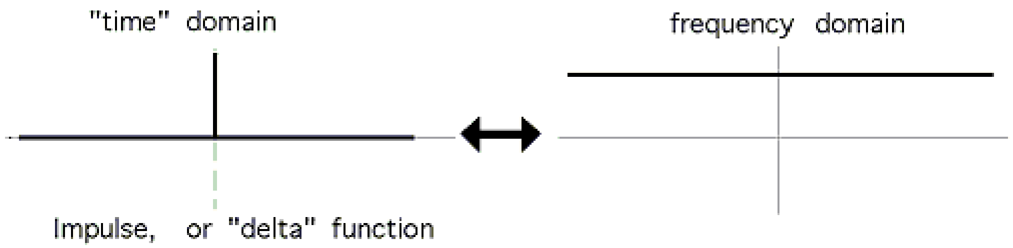
Rule 2: Damping causes spread





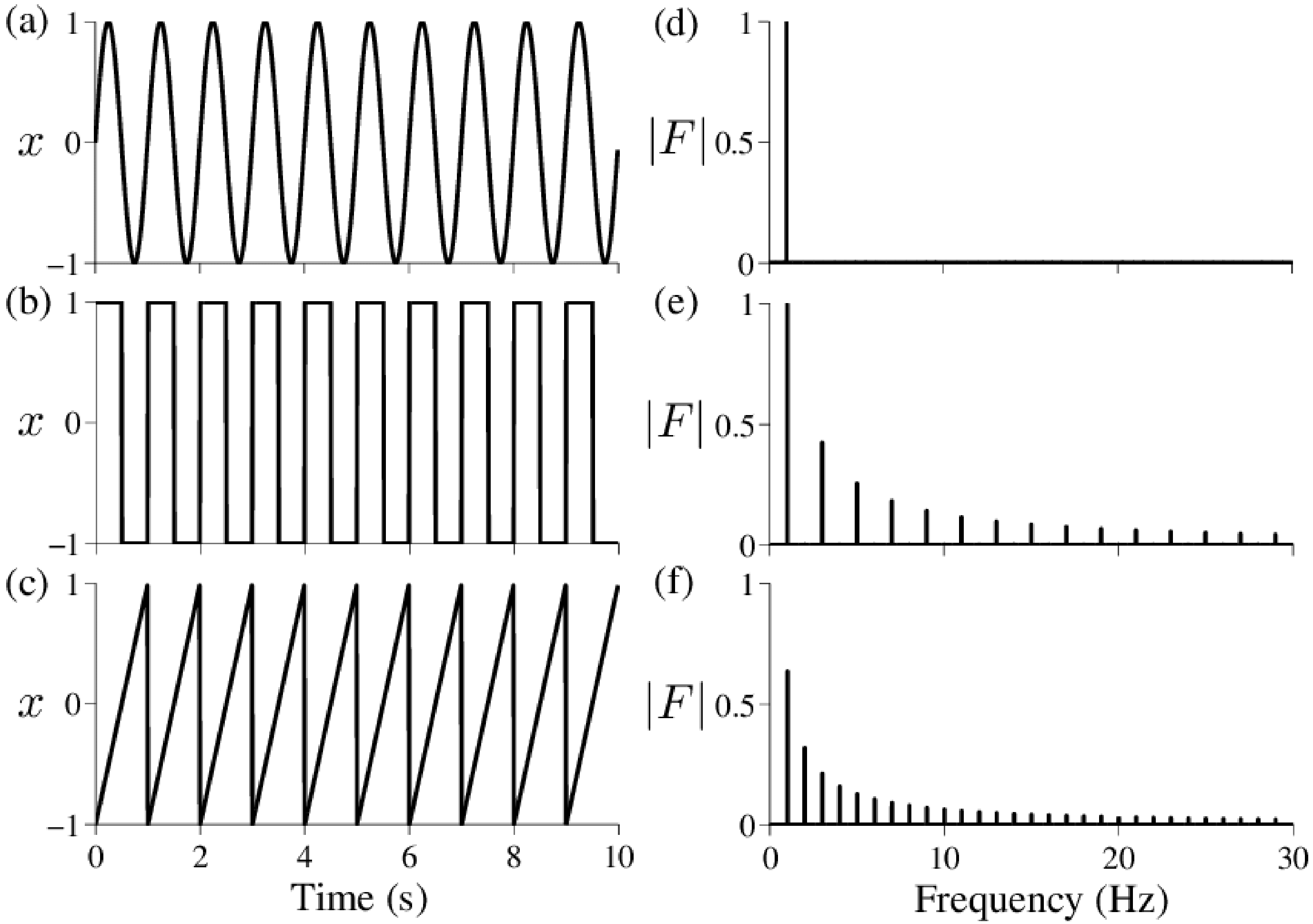
Rule 3: Sharp changes (edges) require a lot of frequencies



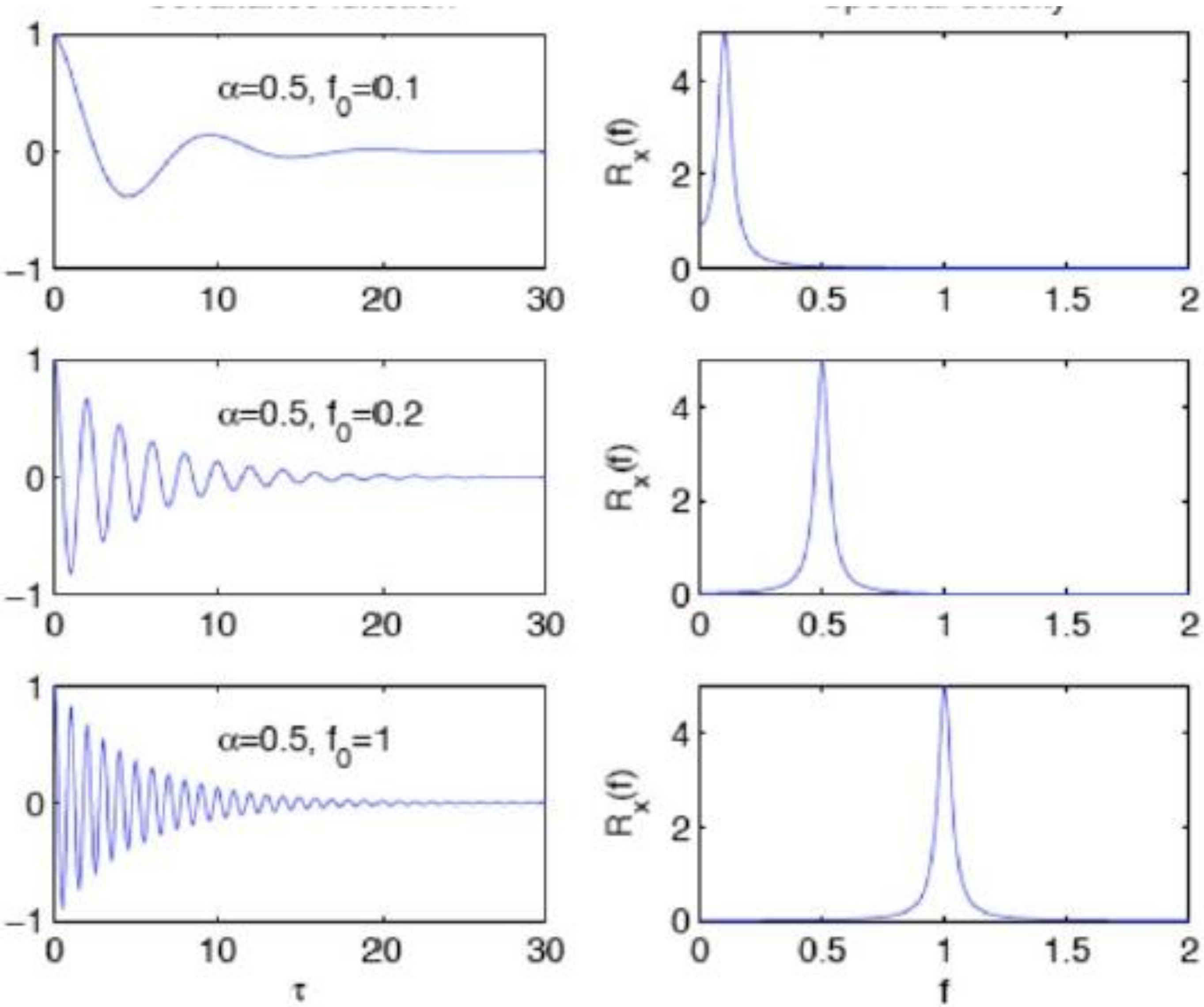


… and an extremely sharp change (impulse) requires ALL the frequencies!!

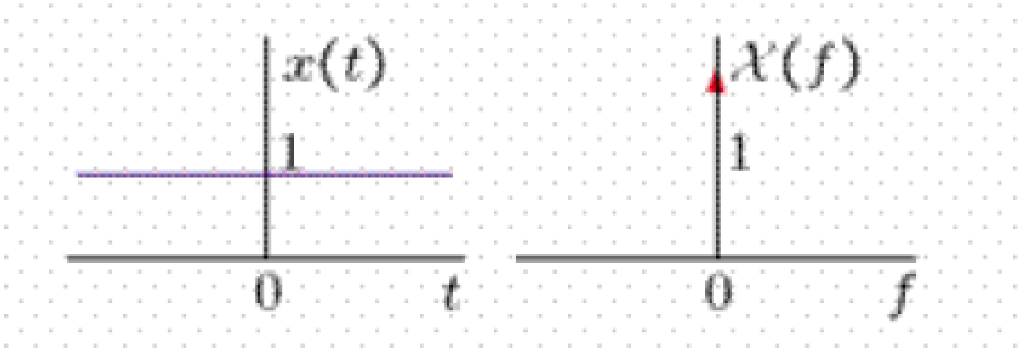
Rule 3: Sharp changes (edges) require a lot of frequencies

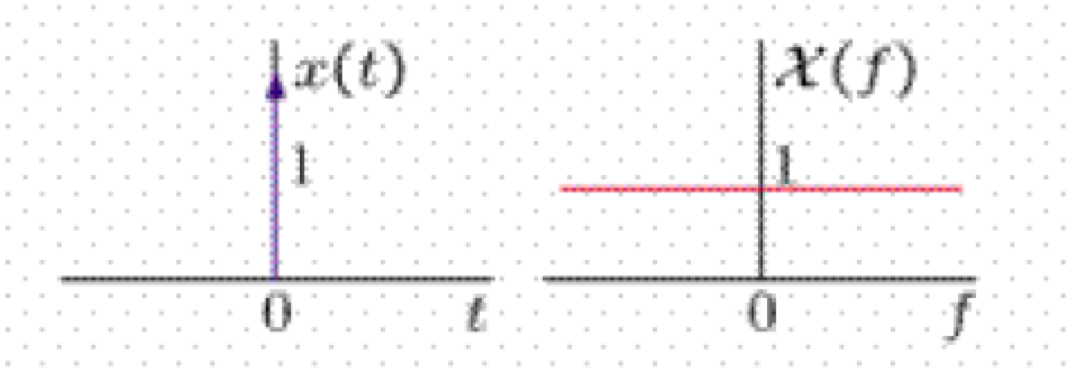


Rule 4: Periodic functions have discrete spectra.



Rule 5: Multiple effects can be combined.





Rule 6: Duality

Questions?? Thoughts??

lec 6 System Basics

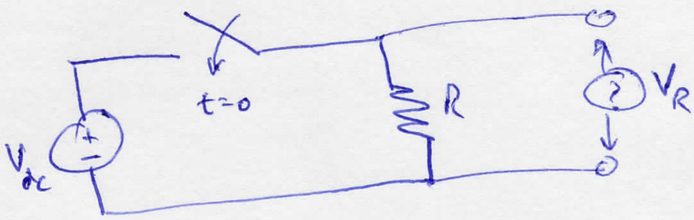
— we have previously looked at signals and several of their types (classifications)

— Signal : collection of data
: carries some information (even noise)

— Today we look at Systems

why? < all around us
we often need to extract or modify info in signals

examples < car
ear
circuit
room



$V_R = IR$

— So many Systems!

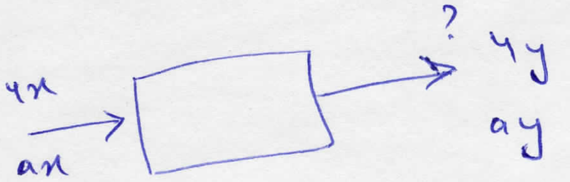
— How can we know response?

— Some general way of modeling it? (independent of system ~~nature~~ field)

— Perhaps we can study some properties?

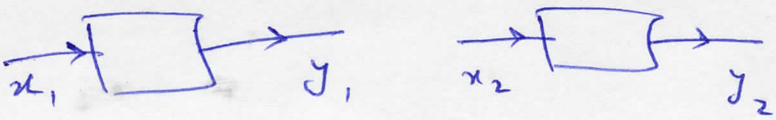
→ Probe a block box?

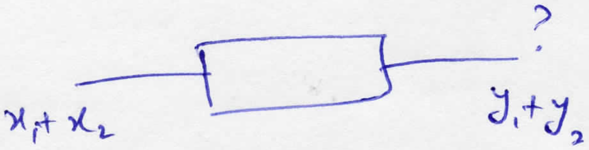
Q. → What happens if I scale the input?



→ if output also scaled by same factor } ← Homogenous

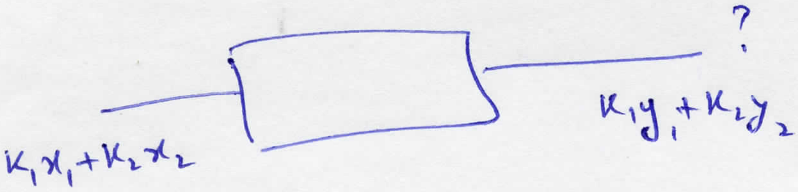
Q. → what happens if I add signals? (inputs)





→ if output also sum of original outputs } ← Additive.

→ we can combine the above two in one form



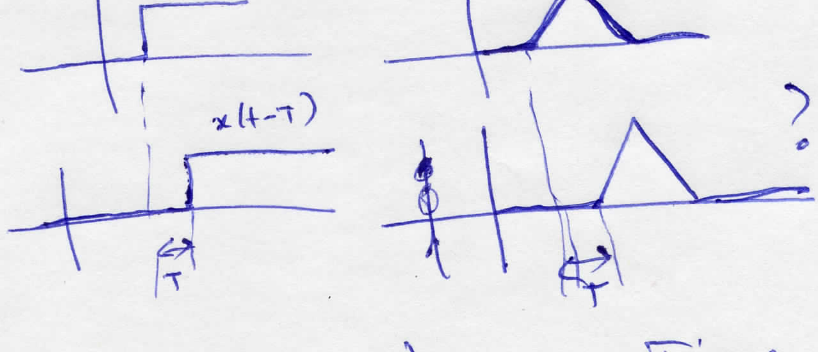
Homogeneous
Additive.

→ if yes, then system allows Superposition.

→ And we call such a system "Linear".

→ huge fields < linear / non linear

Q. → what happens when input is shifted?

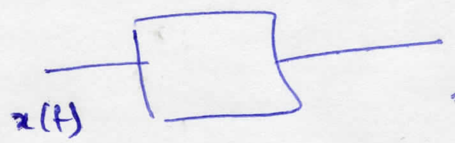


→ if output is also shifted by same amount } ← Time Invariant

→ ~~Huge~~ classification $<$ TI / TV

→ Huge class : LTI

Q. → Does system output depend on past or future inputs? (or only on current input?)



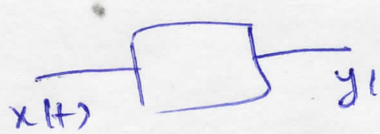
$y(t) = f(x(t))$?

$y(t) = f(x(t), x(t \pm T))$? $\quad T \neq 0$

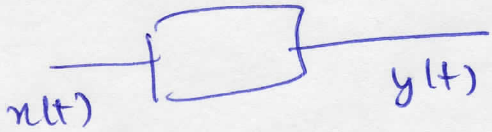
→ if output at time t depends only on $x(t)$ (and not on $x(t \pm T)$ etc.) then → Memoryless

(also called Instantaneous.)

→ otherwise : Dynamic (or Memory system)



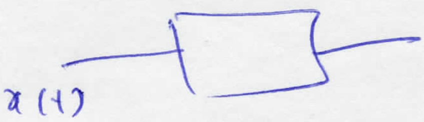
$y(t) = 4x(t) + x^2(t)$ ← Memoryless



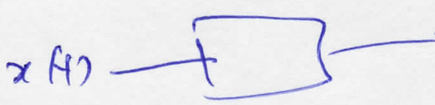
$y(t) = 4x(t) + x^2(t+1)$ ← dynamic

Q. → Does output depend only on current + past inputs?

→ if yes : Causal (all real systems are causal) ← cannot have output before even applying input.



$y(t) = 2x(t) + 4x(t-1)$ ← causal



$y(t) = 2x(t) + 4x(t+1)$ ← noncausal

→ why study non-causal?

— suppose signal is recorded!

→ then future and past values are available

$\{x(t_1)\ x(t_2)\ x(t_3)\ x(t_4)\ x(t_5)\}$

$$y(t_1) = \frac{x(t_1)+x(t_2)+x(t_3)+x(t_4)+x(t_5)}{5}$$

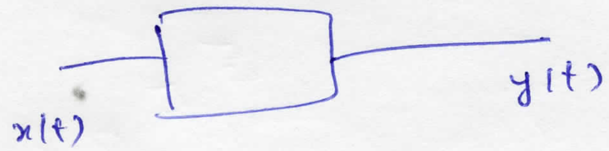
etc.

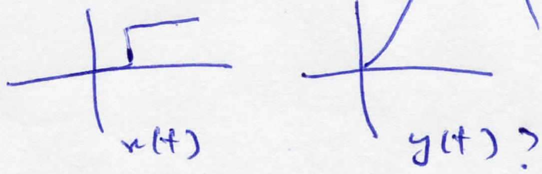
→ or independent variable can be non-time

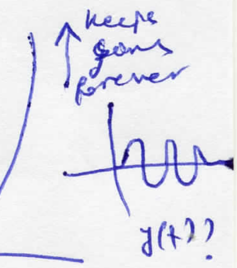
$x(l_1)\ x(l_2)\ x(l_3)\ x(l_4)$ Temperature on wire

$$y = \frac{x(l_1)+x(l_2)}{2}$$

Q: → Do Bounded inputs give bounded output?





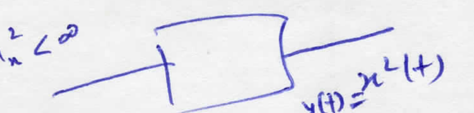


→ if no: system unstable (BIBO-unstable)

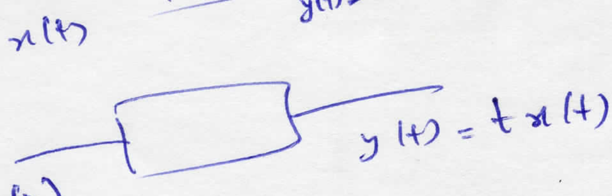
→ if yes: system stable (BIBO-stable)

$|x(t)| \le M_x < \infty$

$|x^2(t)| = |x(t)|^2 \le M_x^2 < \infty$



$y(t) = x^2(t)$ ← ~~Bond~~ BIBO stable



$y(t) = t\,x(t)$ ← BIBO unstable

e.g. set $x(t) = u(t)$

Q → Are inputs outputs continuous or discrete?

CT System → both input/output CT

DT system → ______ DT

C/D system → input: CT, output DT

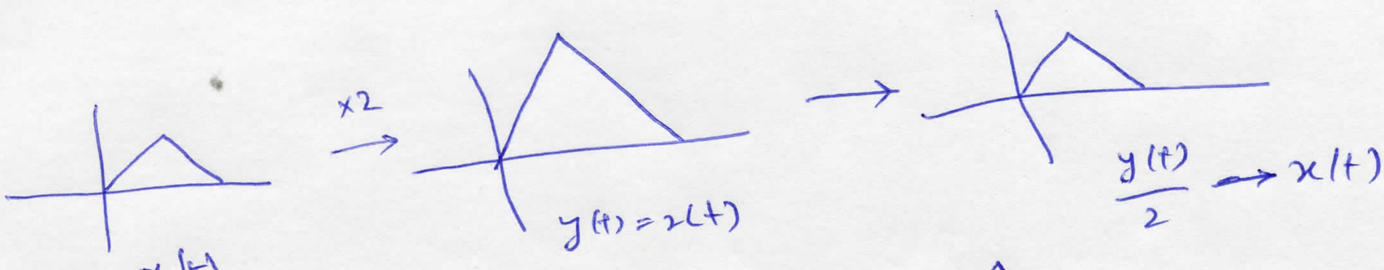
D/C system →

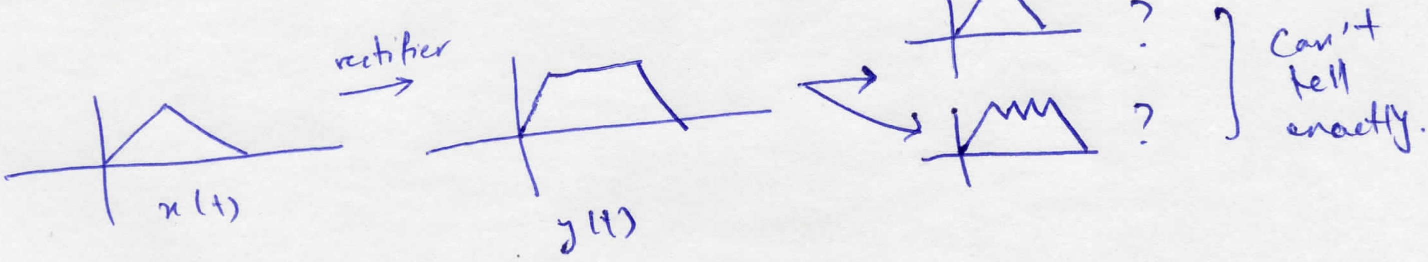
Q. Are inputs/outputs → Analog or digital?

- Analog (both analog)
- Digital (both digital
- ADC
- DAC

Q. → Can we recover the input exactly from the output? (without knowing input of course)

if yes → Invertible.



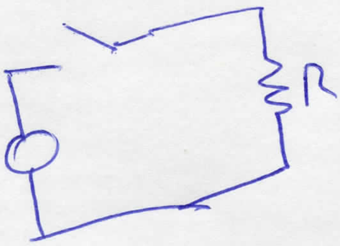


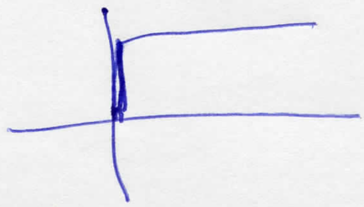
→ SISO, MIMO etc.

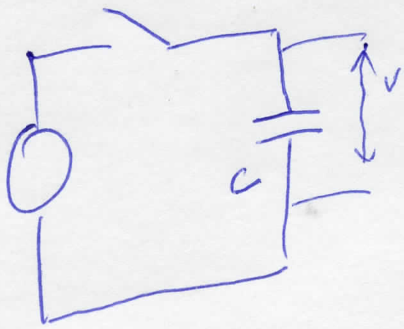
→ Linear System response breakdown

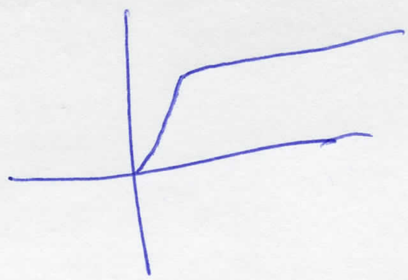
Total Resp = Zero-input resp + Zero state response (Zero state = zero initial condition)

V=IR

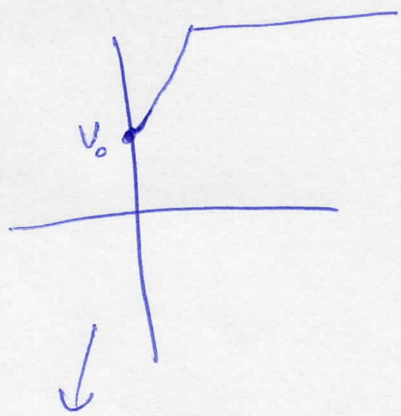


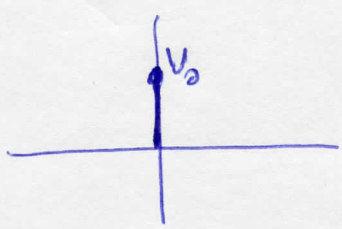






or





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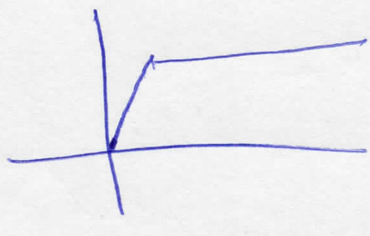


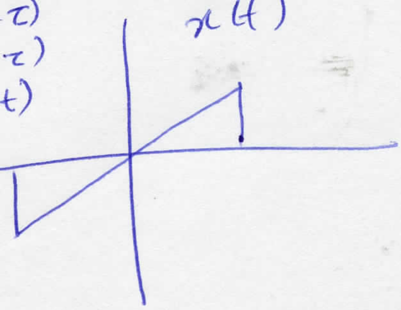
Zero input resp + Zero-state response

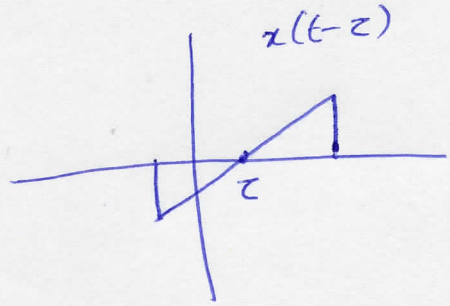
Lec 7 Practice Problems (Signal & System basics).

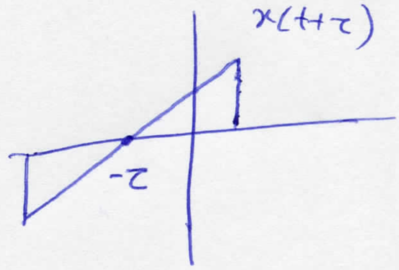
1

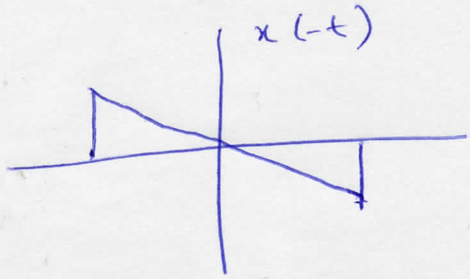
Given $x(t)$

Identify $x(t+\tau)$
$x(t-\tau)$
$x(-t)$









2 Given $x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{other} \end{cases}$

Find $x(t)$ delayed by 2, advanced by 2, and reflected

→ delayed by 2 is $x(t-2)$

$$x(t-2) = \begin{cases} 1 & 0 \le t-2 \le 1 \ \text{or} \ 2 \le t \le 3 \\ 0 & \text{o/w} \end{cases}$$

→ advanced by 2 is $x(t+2)$

$$x(t+2) = \begin{cases} 1 & 0 \le t+2 \le 1 \ \text{or} \ -2 \le t \le -1 \\ 0 & \text{o/w} \end{cases}$$

→ reflected is $x(-t)$

$$x(-t) = \begin{cases} 1 & 0 \le -t \le 1 \ \text{or} \ -1 \le t \le 0 \\ 0 & \text{o/w} \end{cases}$$

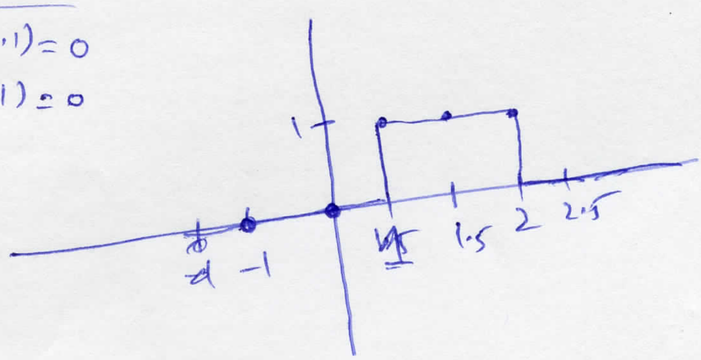
(3) Tables can also be helpful for mixture of operations!

Given $x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{o/w} \end{cases}$ find $x(-t+2)$

t	$x(-t+2)$
-1	$x(3) = 0$
0	$x(2) = 0$
1	$x(1) = 1$
1.5	$x(0.5) = 1$
2	$x(0) = 1$
2.5	$x(-0.5) = 0$
2.1	$x(-0.1) = 0$
0.9	$x(1.1) = 0$

further check

t	$x(-t+2)$
-1	0
0	0
1	1
1.5	1
2	1
2.5	0



(4) If T_0 is a period of a function $x(t)$ then so is $T_k = kT_0$ for k integer. Prove it. Useful result!

Given is: $x(t+T_0) = x(t)$ for all t

let $t_1 = t + T_0$, then by periodicity

$$x(t_1 + T_0) = x(t_1)$$

$$x(t + T_0 + T_0) = x(t+T_0) \overset{\text{by}}{=} x(t)$$

$\Rightarrow$ $x(t+2T_0) = x(t)$ $\Rightarrow$ so $2T_0$ is also a period

let $t_2 = t + 2T_0$ and repeat!

Problem

4. Prove that the fundamental period of $\cos(wt)$ is $T_0 = 2\pi/w$. $-\infty \leq t \leq \infty$

→ let $x(t) = \cos(wt)$

→ let us assume there is a non-zero T_0 such that

$$x(t+T_0) = x(t) \quad \forall t$$

$$\Rightarrow \cos(w(t+T_0)) = \cos(wt) \text{ ——— } \star$$

$$\cos(wt + wT_0) = \cos(wt)$$

How is that possible with non-zero T_0?

→ Trigonometry comes to out help!

$$\cos(\theta_t + 2\pi) = \cos(\theta_t)$$

$\star$ holds if $wT_0 = 2\pi \Rightarrow T_0 = 2\pi/w$

$$\Rightarrow w = \frac{2\pi}{T_0} = 2\pi f_0$$

→ From now on we will simply say that fundamental period of $\cos(wt)$ ~~is~~ can be calculated from $w = \frac{2\pi}{T_0}$. E.g. $\cos(2t) \Rightarrow 2 = \frac{2\pi}{T_0}$

$$\Rightarrow T_0 = \pi$$

→ Above also holds for $\cos(wt+\theta)$, $\sin(wt)$, $\sin(wt+\theta)$, and e^{jwt}.

$$\hookrightarrow e^{jwt} = \underbrace{\cos(wt)}_{w = \frac{2\pi}{T_0}} + j\,\underbrace{\sin(wt)}_{w = \frac{2\pi}{T_0}}$$

Problem 5 Find energy & power of

$$z(t) = \begin{cases} 1 & 0 \le t \le 10 \\ 0 & \text{otherw} \end{cases}$$

$$\rightarrow \quad E_z = \int_{-\infty}^{\infty} |z(t)|^2 dt = \int_0^{10} 1^2 dt = \int_0^{10} dt = t\Big|_0^{10} = 10 - 0 = 10$$

$$\rightarrow P_z = \underbrace{\lim_{T\to\infty} \frac{1}{T}}_{T\to\infty} \underbrace{\int_{-T/2}^{T/2} |z(t)|^2 dt}_{\text{as } T\to\infty \text{ this approach } E_z = 10}$$

→ Dividing a finite value with infinite value

$$\Rightarrow P_z = 0$$

Problem 6 characterize $x(t) = \sqrt{2}\cos(\pi t/2 + \pi/4) \quad -\infty \le t \le \infty$

→ Analog? Yes , → Deterministic? Yes (no random parameters)

→ Periodic? Yes, with f. Period given by $w = \frac{2\pi}{T_0} \Rightarrow \frac{\pi}{2} = \frac{2\pi}{T_0}$

$\Rightarrow \pi T_0 = 4\pi \Rightarrow T_0 = 4$ (hint by $t = T_0$ in original and see it becomes 2π)

→ even? $x(t) \overset{?}{=} x(-t)$

$\sqrt{2}\cos\left(-\frac{\pi t}{2} + \frac{\pi}{4}\right)$

(5)

Problem (7) even or odd or neither?

(a) $\cos(t)$?

even : $x(t) = x(-t)$
odd : $x(t) = -x(-t)$

$\cos(-t) = \cos(t)$ ← even! (for trigonometry $\cos(\theta) = \cos(-\theta)$)

(b) $\sin(t)$?

$\sin(-\theta) = -\sin(\theta)$

$\Rightarrow \sin(t) = -\sin(-t)$ ← odd.

(c) $x(t) = -10t$

check $x(t) \overset{?}{=} x(-t) \Rightarrow -10t \overset{?}{=} -10(-t) = 10t$ X $\boxed{\text{not even}}$

check $x(t) = -x(-t) \Rightarrow -10t \overset{?}{=} -(-10(-t)) = -10t$ ✓ $\boxed{\text{odd}}$

Problem (8) Simplify $y(t) = \frac{1}{2}\int_{-\infty}^{\infty} x(\tau)\left(\delta(\tau-4) + \delta(\tau+4)\right) d\tau$

Recall $\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau)$ ← so $\delta(t-\tau)$ picks the value of x for $t-\tau=0$ $\Rightarrow t=\tau$

$$y(t) = \frac{1}{2}\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau-4)\,d\tau + \frac{1}{2}\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau+4)\,d\tau$$

$$= \frac{1}{2}x(4) + \frac{1}{2}x(-4)$$

Problem 8 characterize the system $y(t) = e^{tx(t)}$

1. Memoryless? Yes
2. Invertible? No $\quad x(t) = \dfrac{\ln(y(t))}{t}$ undefined at $t=0$.
3. causal? Yes
4. stable? NO
5. TI? NO. $\quad y_{\bullet}(t)\Big|_{x(t-t_0)} \stackrel{?}{=} y(t-t_0)$

$$e^{(t\,x(t-t_0))} \stackrel{?}{=} e^{((t-t_0)\,x(t-t_0))} \quad \times \text{ No.}$$

6. Linear? No.

Lec 8 DT Signals & Systems
Time Domain

Q. what is a DT system? Input & output DT signals!

- so let's talk about DT signals a while
- repeat some of previous but focusing on DT.

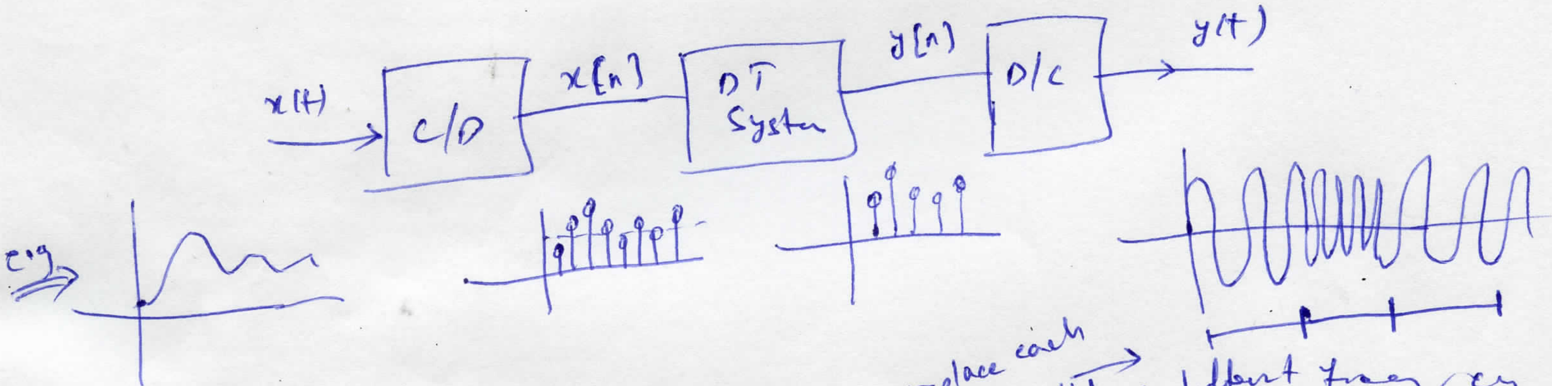
$t \rightarrow n$ in place of t we shall use notation with n

$x[n] \rightarrow n$ discrete e.g.
- n = integers
- n = even integer
- $n = 0.5k$ k integers

} uniformly spaced

- $n = 0, 0.1, 0.5, 1, 10, 100$ } non-uniformly spaced

C-D-D-C → we often discretize CT signals for easy processing



replace each sample with a sinusoid of different frequency.

Energy & Power

$x(t)$ $\quad E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$

$x[n]$ $\quad E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2$

if $x[n]$ periodic i.e. $x[n] = x[n+N_0] \Rightarrow P_x = \frac{1}{N_0} \sum_{\text{one period}} |x[n]|^2$

Shifting & reversal

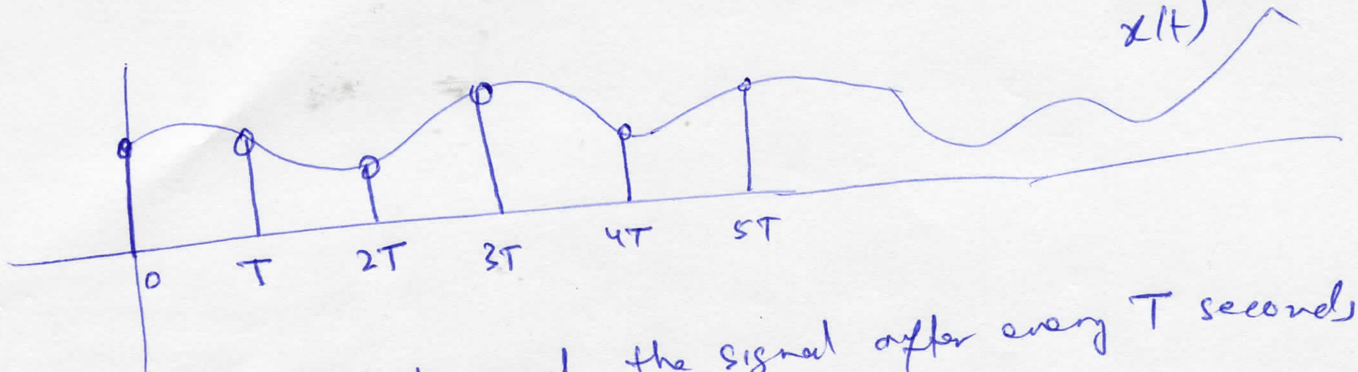
$x_d[n] = x[n-M]$ $\quad M$ +ve integer

$x_a[n] = x[n+M]$ $\quad$ "

$x_r[n] = x[-n] \rightarrow$ horizontal flip

$x_v[n] = -x[n] \rightarrow$ vertical flip

<u>sampling</u> → our way of getting DT from CT!



→ let's say I want to <u>read</u> the signal after every T seconds.

→ my readings look like this

$$\left[x(t=0) \quad x(t=T) \quad x(t=2T) \quad x(t=3T) \quad x(t=4T) \;\ldots\; x(t=nT) \right]$$

→ for simplicity I will replace with DT notation + Addition infor of sampling rate (T)

$$\left[x[0] \quad x[1] \quad x[2] \quad x[3] \quad x[4] \;\ldots\; x[n] \ldots \right]$$

DT version of $x(t)$ →

& T given seperately eg $T = 0.5$ sec. ← sampling rate.

Simple Uniform Sampling formula → $x[n] = x(t=nT)$

e.g. $x(t) = 4t+5$
sample at $T = 0.5$ sec.
then sixth sample is
$x[6] = x(t=6T)$
$= x(3) = 12+5 = 17$

<u>Downsampling</u> (compression) → let's say you don't want to have samples every 0.5 sec but to every 1 sec.

$$x[n] \rightarrow \left[x[0] \quad \cancel{x[1]} \quad x[2] \quad \cancel{x[3]} \quad x[4] \ldots \right]$$

$$x_d[n] \rightarrow \left[x_d[0] \quad x_d[1] \quad x_d[2] \ldots \right]$$

$x_d[n] = x[2n]$ ← Picked every second sample from DT $x[n]$

← So you've Downsampled by a factor of 2!

→ In general, you can downsample by a factor M as

$$x_d[n] = x[Mn] \quad M \text{ integer.}$$

→ What if you wanted to have extra samples in between?

step 1 → upsample (expand)

$$x[n] \rightarrow \left[\, x[0] \;\; x[1] \;\; x[2] \;\; x[3] \;\; x[4] \;\ldots \right]$$

$$x_u[n] \rightarrow \left[\, x_u[0] \;\; 0 \;\; x_u[2] \;\; 0 \;\; x_u[4] \;\; 0 \;\; x_u[6] \;\ldots \right]$$

or ⇒ $$x_u[n] = \begin{cases} x[n/2] & n = 0, \pm 2, \pm 4 \\ 0 & \text{o/w} \end{cases}$$

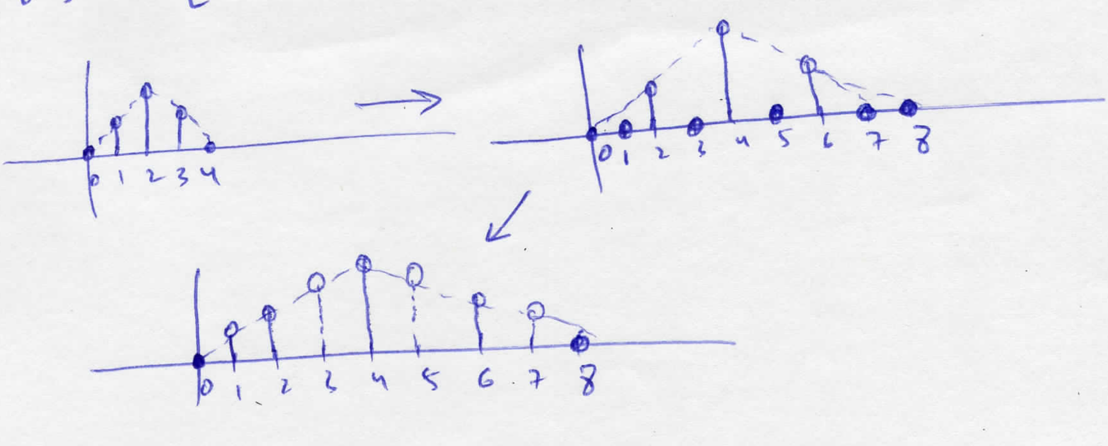
for general upsampling by factor L we have

$$x_u[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{o/w} \end{cases}$$

step 2 replace the inserted zeros by Interpolations.

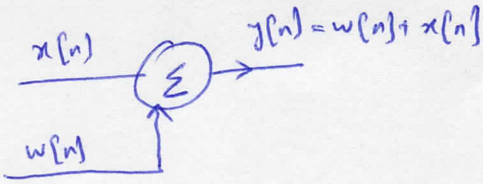
e.g. $$x_i[n] \rightarrow \left[\, x_u[0] \;\; \underset{n=1}{\frac{x_u[0]+x_u[2]}{2}} \;\; \underset{n=2}{x_u[2]} \;\; \underset{n=3}{\frac{x_u[2]+x_u[4]}{2}} \;\; \underset{n=4}{x_u[4]} \;\ldots \; n=5 \ldots \right]$$

→ Graphically



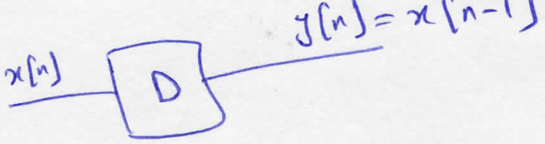
Denoting DT operations through blocks

→ Addition



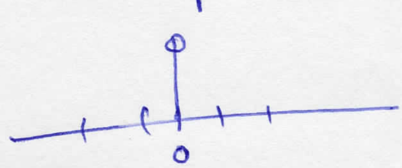
→ Scaling $\quad x[n] \xrightarrow{\ a\ } y[n] = a\,x[n]$

→ Delay



Some common DT signal Models

→ DT Delta $\quad \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$



Q: Try to draw shifted versions!

→ DT unit step $\quad u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

→ DT complex exponential $\quad e^{j\Omega n} = \cos(\Omega n) + j\sin(\Omega n)$ where Ω = angular freq

→ link b/w Ω and ω. If a CT sinusoid is sampled every T seconds, we can link the two as

$$\cos(\omega t) \xrightarrow[\text{rate } T]{\text{sample at}} \cos(\Omega n)$$

$\left(\begin{array}{l} \cos(\omega t) \\ \to \cos(\omega n T) \\ \to \cos(\omega T n) \\ \to \cos(\Omega n) \end{array}\right)$

with $\Omega = \omega T$

→ DT exponential

→ we had the CT exponential e^{at}

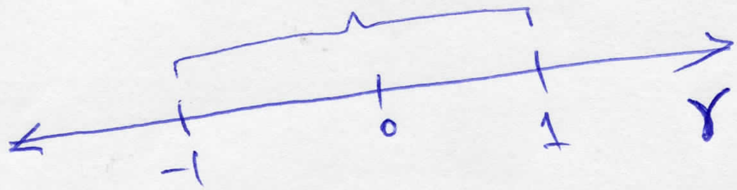
- $a > 0$ growing exponentially
- $a = 0$ constant
- $a < 0$ decaying exponential

→ for DT we have γ^n

Q. Given $n = 0, \pm 1, \pm 2, 3 \cdots$

is γ^n growing, constant (in magnitude), or decaying?

→ If γ is real then



$|\gamma|$ — <1 Decaying; $=1$ neither; >1 growing

e.g.

$(0 \cdot 5)^n$ ← ~~growing~~ decay

$(-0 \cdot 5)^n$ ← changing sign but decaying

$(1 \cdot 5)^n$ ← grow

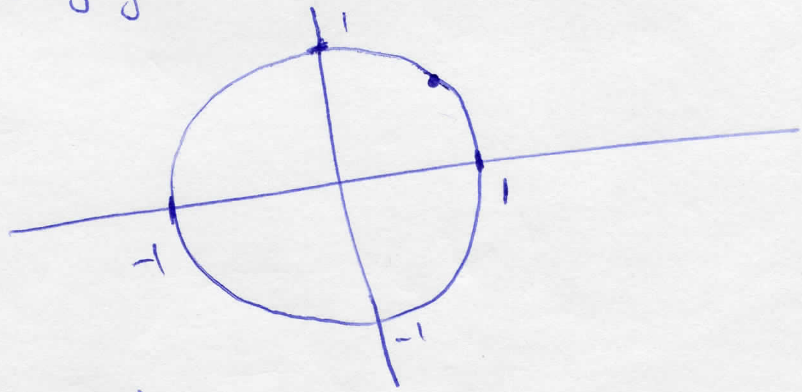
$(1)^n$ ← neither.

→ Same holds for complex γ!

$\gamma = a + jb \Rightarrow |\gamma| = \sqrt{a^2 + b^2}$

$|\gamma| < 1$ Decaying etc.

Q. what does this mean in complex plain



Inside unit circle → decaying

outside unit circle → growing

on unit circle → neither

Lec 9 Convolution & IR

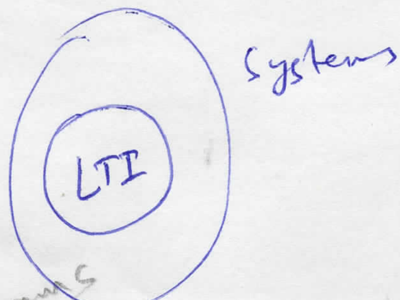
Q: How to find the output of an arbitrary system to an arbitrary input?

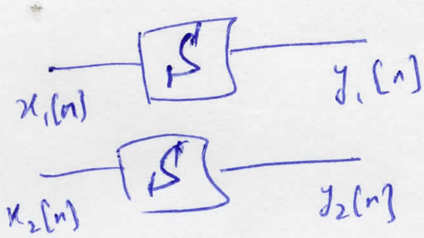
Problem
- ⓐ So many types of systems!
- ⓑ So many signals! (input candidates)

ⓐ ⇒ Perhaps consider first a subclass of systems?

ⓑ ⇒ Perhaps consider a simple model that can cover all signals?

ⓐ ⇒





see also block diagrams on next page

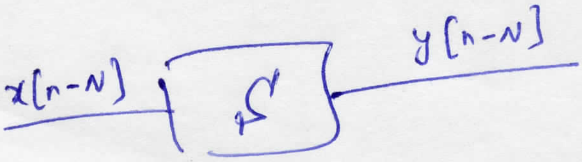
Linear ⟶

In other words ⇒ for linear system if $S(x_1[n]) = y_1[n]$

and $S(x_2[n]) = y_2[n]$ then $S(ax_1[n]+bx_2[n]) = aS[x_1[n]] + bS[x_2[n]]$

$= ay_1[n] + by_2[n]$

TI



In other words: for TI system if $S(x[n]) = y[n]$

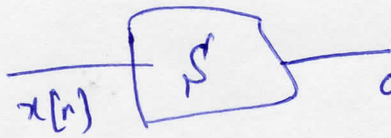
then $S(x[n-N]) = y[n-N]$

e.g. are these LTI? (i) $\mathcal{S}[x(n)] = a\,x[n]$

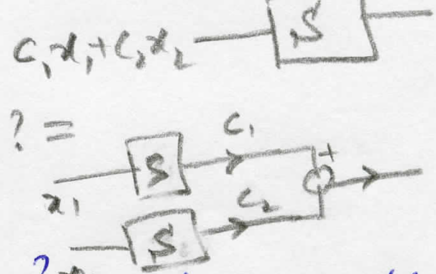
(ii) $\mathcal{S}[x(n)] = a x[n] + b$

(iii) $\mathcal{S}[x[n]] = a\,x(n) + bn$

Let's see (i)

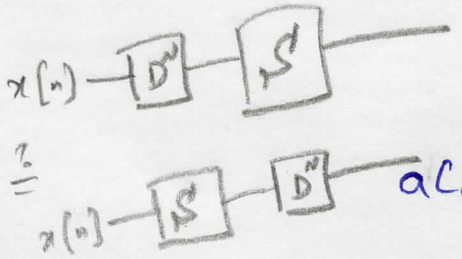


?=



Linearity test $\mathcal{S}[c_1 x_1(n) + c_2 x_2(n)] \overset{?}{=} c_1 \mathcal{S}(x_1(n)) + c_2 \mathcal{S}(x_2(n))$

$a c_1 x_1(n) + a c_2 x_2(n) \overset{\checkmark}{=} c_1 a x_1(n) + c_2 a x_2(n)$



TI test $\mathcal{S}(\underset{\text{input delayed}}{x(n-N)}) \overset{?}{=} \underset{\text{output delay}}{y[n-N]}$ with ~~$y[n] = \mathcal{S}(x(n))$~~ $\rightarrow$ with $y(n) = a x[n]$

$a\,x(n-N) \overset{\checkmark}{=} a\,x[n-N]$

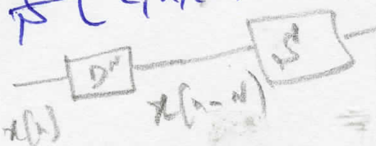
(ii) $\mathcal{S}[c_1 x_1(n) + c_2 x_2(n)] \overset{?}{=} c_1 \mathcal{S}(x_1(n)) + c_2 \mathcal{S}(x_2(n))$

not linear $a c_1 x_1(n) + a c_2 x_2(n) + b \overset{\times}{=} c_1 a x_1(n) + c_1 b + c_2 a x_2(n) + c_2 b$

TI ✓ $\mathcal{S}(x(n-N)) \overset{?}{=} y[n-N] \rightarrow$ with $y[n] = a x[n] + b$

$a\,x[n-N] + b \overset{\checkmark}{=} a\,x[n-N] + b$

(iii) $\cancel{S'(c_1 x_1(n) + c_2 x_2(n))}$ not linear (easy to see)



$ax(n-N) + bn \rightarrow$ not necessarily taken from $x[n]$'s n! (can't prove it!)

not TI

$$S'(x(n-N)) \stackrel{?}{=} y(n-N) \longrightarrow \text{where } y(n) = ax[n] + bn$$

$$ax(n-N) + b\underline{n} = ax(n-N) + b(n-N)$$

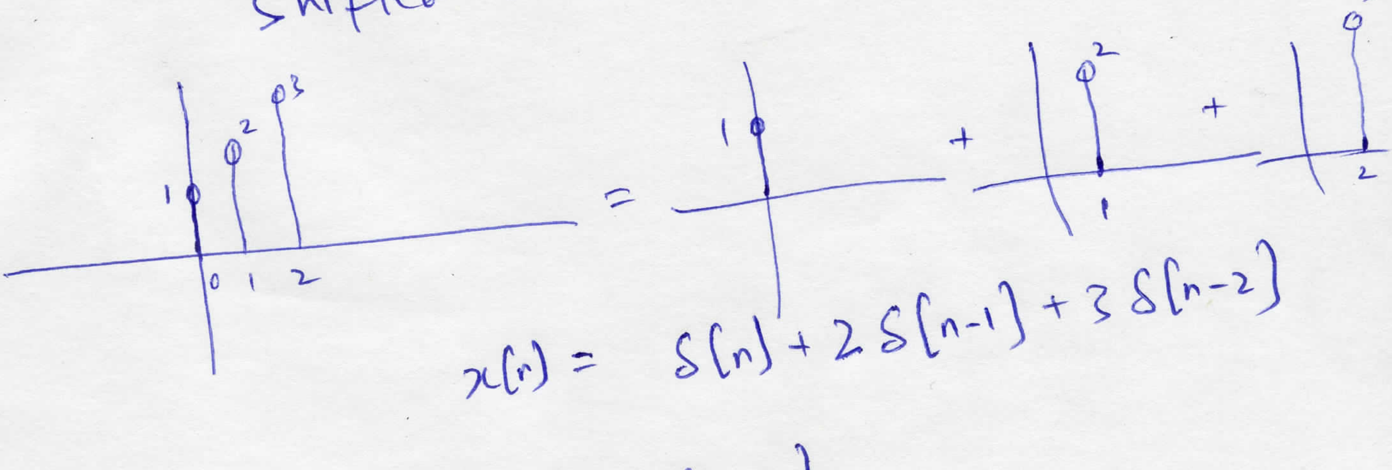
not linked to input, so won't change

→ o.k, so let's stick to LTI for convenience

→ Now let's find a generic signal Model.

claim: Any DT signal can be written as a

Sum of
Scaled and
Shifted Impulses!



$$x(n) = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

$$x[n] = \sum_{\forall k} x[k]\,\delta[n-k]$$

→ now let's bring these two together!

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

← let's call it "Impulse Response".

$$a_0\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow a_0 h[n]$$

$$\cancel{a_0\delta[n] +} a_1\delta[n-1] \longrightarrow \boxed{\text{LTI}} \longrightarrow a_1 h[n-1]$$

$$\underset{\downarrow \atop x[0]}{a_0}\delta[n] + \underset{\downarrow \atop x[1]}{a_1}\delta[n-1] \longrightarrow \boxed{} \longrightarrow a_0 h[n] + a_1 h[n-1]$$

$$\underbrace{\sum_{\forall k} x(k)\,\delta(n-k)}_{x[n]} \longrightarrow \boxed{\text{LTI}} \longrightarrow \underbrace{\sum_{\forall k} x(k)\,h(n-k)}_{y[n]}$$

$$y[n] = \sum_{\forall k} x(k)\,h(n-k)$$

$$= x[n] * h[n]$$

$$= \underline{\underline{\text{Convolution}}}.$$

so basically
→ Convolution gives us the output of an LTI system to an arbitrary input signal! (you need to know only the signal and IR)

5

→ So two take aways — Convolution ← (mathematical operator, helps find LTI response)

— IR $(h[n])$ ← can fully characterize an LTI system (as we shall see soon)

→ both of these need full attention!

→ useful convolution properties

<u>identity</u> $x[n] * \delta[n] = x[n]$

<u>Commutative</u> $x_1[n] * x_2[n] = x_2[n] * x_1[n]$

so $\sum_{\forall k} x[k]\,h[n-k] = \sum_{\forall k} h[k]\,x[n-k]$

<u>Distributive</u> $x_1[n] * \left(x_2[n] + x_3[n]\right) = \left[x_1[n] * x_2(n)\right] + \left[x_1[n] * x_3[n]\right]$

<u>Associative</u> $x_1[n] * \left(x_2(n) * x_3(n)\right) = \left(x_1[n] * x_2[n]\right) * x_3[n]$

<u>Shifting</u> if $x_1[n] * x_2[n] = c[n]$

then $x_1[n-m] * x_2[n-p] = c[n-m-p]$

→ like Integrals, we have convolution tables.

→ also mostly done on computers.

Solving convolution
- Matlab
- by Tables + properties
- graphically
  - By Plots
  - sliding tape.

1. Matlab $x = [-1\ \ 0\ \ 1]$, $h = [0\ \ 1\ \ 0]$

   $y = \text{conv}(x, h)$ ← command.

2. $u[n] * (\delta[n] + 5u[n])$?

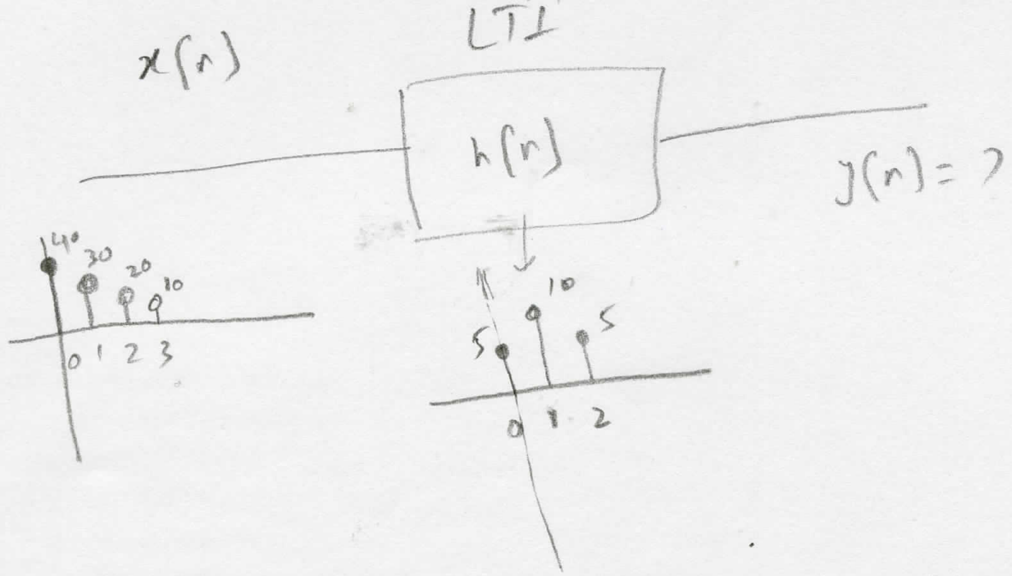
   by Dist. prop. ⇒ $\underbrace{(u[n] * \delta[n])} + u[n] * 5u[n]$

   by table → $u[n] + 5[(n+1)u[n]]$

3. By plots & By tape similar (same basic idea)

   ↓

   easier to draw

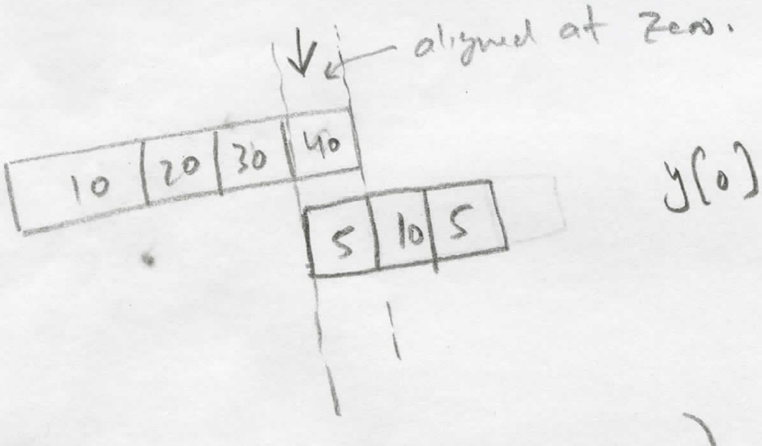


$x[-k+n]$

$$y[n] = x[n] * h[n] = \sum_{\forall k} x[k]\,h[n-k] = \sum_{\forall k} h[k]\,x[n-k]$$

$$\Rightarrow y[0] = \sum_{\forall k} h[k]\,x[-k]$$ ← input flipped

$$\Rightarrow y[1] = \sum_{\forall k} h[k]\,x[-k+1]$$ ← flipped & shifted



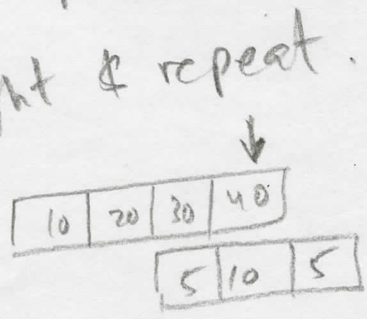
$y[0] = 5 \times 40 = 200$

Step 1 Flip the input (or IR)

Step 2 Align input & IR at zero.

Step 3 Multiply overlapping cells & add results

Step 4 Shif right & repeat.



$y[1] = 30 \times 5 + 40 \times 10$
$= 150 + 400 = 550$

Lec 10 IR (LTI) + Zero input response + practice
+ Lec 11

Recall → IR = response of a system to an Impulse

For LTI → IR can fully characterize an LTI system!

- → gives output (through convolution)
- → can be used to tell if system causal / stable / Dynamic

LTI output → for input $x[n]$ & IR $h[n]$ output $y[n] = x[n] * h[n]$

$$= \sum_{\forall k} x[k]\,h[n-k]$$

Visualizing the IR

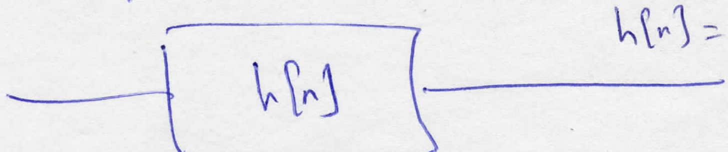
What does an IR ($h[n]$) really say?

or what does $h[n] = [\underset{\uparrow}{2}\ 4\ 8]$ really mean?

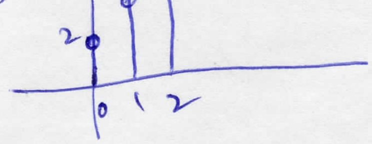
or what is really going on inside the system with $h[n]$ drawn as?

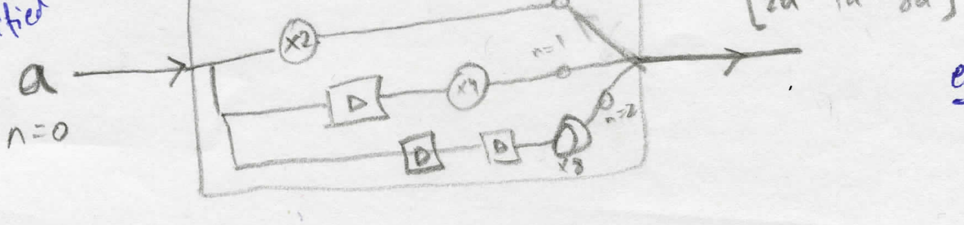
note can be written in $\delta[n]$ as

$h[n] = 2\delta[n] + 4\delta[n-1] + 8\delta[n-2]$



Says: if you give me one value, I will give you three values one after the other" (at specified time intervals) ← h[n]



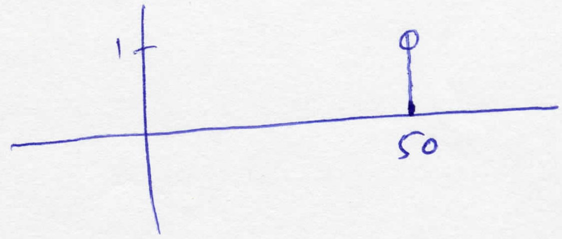


fun: Pick three students, give them a number, ask to multiply by 2, 4, 8 one by one

extra ~~what happens~~

Note if we send b one time unit after a then outputs will start overlapping leading to "convolution"!

Q: what does this IR say?



→ Using h[n] to characterize LTI system

Let's write out the output:

$$y[n] = x[n] * h[n] = \sum_{\forall k} h[k]\, x[n-k]$$

$$y[n] = \underbrace{\dots h[-2]x[n+2] + h[-1]x[n+1]}_{\text{future}} + h[0]x[n] + \underbrace{h[1]x[n-1] + h[2]x[n-2] + \dots}_{\text{past}}$$

Q: when is a system instantaneous (Memoryless)?

→ when output depends only on current input (but not on past or future inputs)

$$\boxed{h[n] = \begin{cases} \neq 0 & n=0 \\ 0 & \text{otherwise} \end{cases}}$$

or $h[n] = K\delta[n]$, K: some constant

Q: from eqn. above what condition should h[n] follow to make system memoryless?

$$h[n] = 0 \quad n \neq 0$$ ← Instantaneous o/w Dynamic

Q: " " " " Causal? (no dependence on future value)

$$h[n] = 0 \quad n < 0$$

Q. what condition should apply to $h[n]$ for BIBO stability?

→ say $x[n]$ is bounded such that $|x[n]| \leq M_x < \infty$

negative terms can cancel out

→ then $$|y[n]| = \left| \sum_{\forall k} h[k]\, x[n-k] \right|$$

no negative terms.

$$\leq \sum_{\forall k} |h[k]|\, |x[n-k]|$$

$$\leq \sum_{\forall k} |h[k]|\, M_x$$

$$= M_x \sum_{\forall k} |h[k]|$$

$$< \infty \quad \text{iff} \quad \sum_{\forall k} |h[k]| < \infty$$

Q. Studied Matrices? what is an eigenvalue/eigenvector?

$A\underline{x} = \lambda \underline{x}$ Defn.

$$\underline{x} \longrightarrow \boxed{A} \longrightarrow \lambda \underline{x}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \lambda \begin{bmatrix} \\ \\ \end{bmatrix}$$

$\boxed{\text{eigenvector}}$: A vector that goes through a matrix and comes out the same except for some possible scaling.

→ That scaling is $\boxed{\text{eigenvalue}}$

why so important?

→ notice above that huge matrix multiplication simply reduced to scaling!

→ further: suppose matrix A has two eigenvector $\underline{x}_1$ and $\underline{x}_2$ with eigenvalues (scalings) λ_1, λ_2. Then

$$\underline{x}_1 \longrightarrow \boxed{A} \longrightarrow \lambda_1 \underline{x}_1 \qquad \underline{x}_2 \longrightarrow \boxed{A} \longrightarrow \lambda_2 \underline{x}_2$$

→ what if another vector is to be multiplied by A?

$$\underline{y} \longrightarrow \boxed{A} \longrightarrow ? \qquad \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \underline{y} \\ \\ \end{bmatrix} = ?$$

huge multiplication? what if we can write $\underline{y}$ in terms of $\underline{x}_1$ and $\underline{x}_2$? Perhaps

$$\underline{y} = a\underline{x}_1 + b\underline{x}_2 \qquad a\underline{x}_1 + b\underline{x}_2 \longrightarrow \boxed{A} \longrightarrow ?$$

$$a \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \underline{x}_1 \\ \\ \end{bmatrix}}_{\lambda_1 \underline{x}_1} + b \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \underline{x}_2 \\ \\ \end{bmatrix}}_{\lambda_2 \underline{x}_2} = a\lambda_1 \underline{x}_1 + b\lambda_2 \underline{x}_2$$

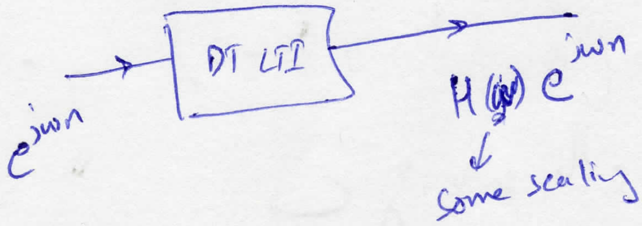
→ so its good to know eigenvectors of a matrix

→ Just like this <u>Systems</u> have <u>eigenfunctions</u>

<u>eigenfunction</u> of a system is a function that comes out of the system unchanged except for some possible scaling.

<u>Q.</u> What are the eigenfunctions of DT LTI Systems?

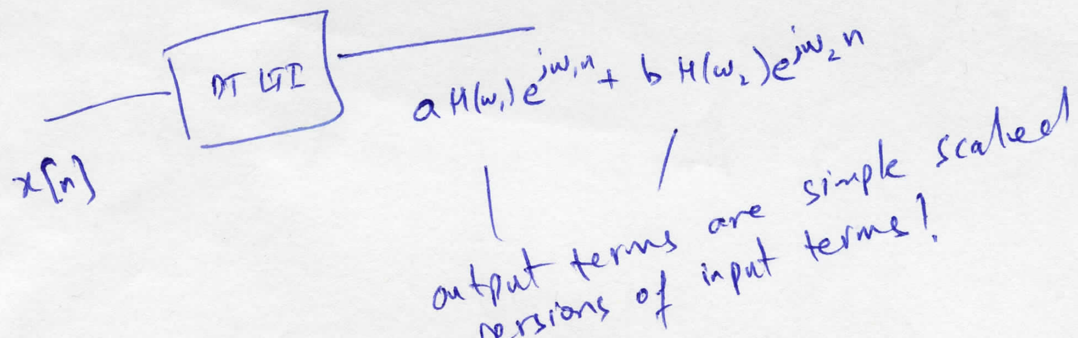
Ans: Complex sinusoids (exponential) are eigenfunctions of DT LTI systems.



Again, why is that useful?

We can often write signal in terms of complex sinusoids, e.g., as $x[n] = a e^{j\omega_1 n} + b e^{j\omega_2 n}$ (toy example)

then



! note → Such rewriting $x[n] = a e^{j\omega_1 n} + b e^{j\omega_2 n} + c e^{j\omega_3 n} \ldots$ we call "Transform". We will see lots of these transforms in this course!

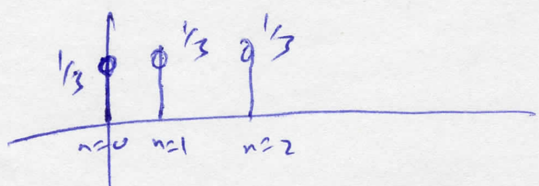
Stay Tuned!!!!...

Example 1

Given an LTI filter with IR

$$h[n] = \frac{1}{3}\left(\delta[n] + \delta[n-1] + \delta[n-2]\right)$$

a) Plot the IR
b) find output to a general input $x[n]$
c) Is it memoryless?
d) causal? e) stable?



a)

b) $y(n) = x[n] * h[n] = \frac{1}{3}\left(x[n] * \delta[n] + x[n] * \delta[n-1] + x[n] * \delta[n-2]\right)$

$$= \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

c) For memoryless $h[n] = 0$ $n \neq 0$, which is not the case here (e.g. $h[1] = h[2] = 1/3$)

d) For causal LTI $h[n] = 0$ $n < 0$, which is the case here!

e) For BIBO stable LTI $\sum_{\forall k} |h[k]| < \infty$

which is the case here!

Example 1

~~Repeat example 1 for~~

$$h[n] = 0.5^n u[n]$$

a) Plot
b) Memoryless?
c) Causal?
d) BIBO stable?

$\sum ar^n$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2$$

Lec 12

1) Eigenfunctions

- Last time we talked about eigenvectors and bases.

$$A\underline{x}_1 = \lambda_1 \underline{x}_1 \quad , \quad A\underline{x}_2 = \lambda_2 \underline{x}_2$$

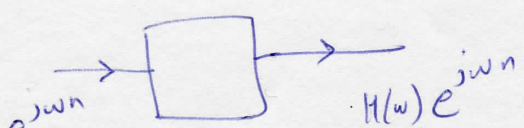
- $\underline{x}_1$ and $\underline{x}_2$ can form bases, so that for any $\underline{x}$ we have

$$\underline{x} = a_1 \underline{x}_1 + a_2 \underline{x}_2$$

- then $$A\underline{x} = a_1 \lambda_1 \underline{x}_1 + a_2 \lambda_2 \underline{x}_2$$

- so all you need to decide or find is
  - (i) what are the bases vectors or functions (eigenfunctions)
  - (ii) what are the weights a_1, a_2
  - (iii) what are the eigenvalues

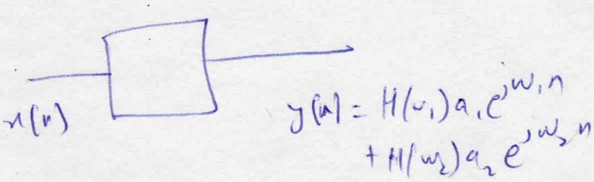
2) Complex Sinusoids are Efs of LTI



eigenvalue for e^{jwn}

- what if we could rewrite general inputs $x[n]$ in the bases functions as complex sinusoids?

$$x[n] = a_1 e^{jw_1 n} + a_2 e^{jw_2 n}$$



3) This is what Z-Transform does essentially

(for DT, for CT Laplace/Fourier do this)

(i) what are the basis functions!

— in general z can be $Ae^{j\phi}$

— mostly we go for $z = e^{jw}$, with bases functions z^n

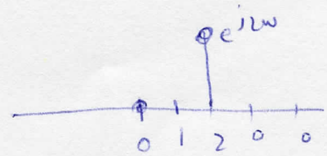
formed as ~~$e^{jw}, e^{j2w}, e^{j3w}, \dots e^{jwn}$~~

(ii) how to find the weights? $(a_1, a_2 \dots)$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

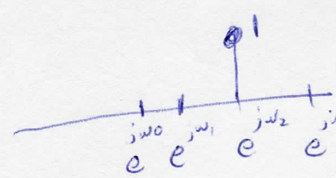
Project $x[n]$ onto bases z^n (projection involves conjugation, so z^{-n})

let's say $x[n] = e^{j2w}\delta[n-2]$ then



$$\underset{0}{x[0]}\, z^{-0} + \underset{0}{x[1]}\, z^{-1} + x[2]\, z^{-2} + \underset{0}{x[3]}\, z^{-3} + \dots$$

$$e^{j2w} e^{-j2w} = 1$$



(iii) Eigenvalues are the Z-transform of Impulse response!

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

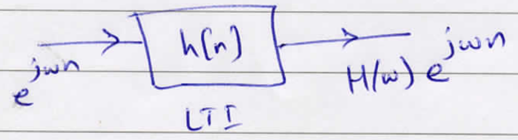
$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n] \Rightarrow X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z) = H(z)X(z)$

Summary $\rightarrow$ If you take Z-Tx of input and IR, then output can be computed as simple multiplication (not convolution)

Lec 13

R

- In previous lecture we talked ~~by~~ abt. complex exponentials being eigenfunctions of DT LTI systems



(and we set $Z^n = e^{j\omega n}$)

- In fact, for DT LTI general everlasting (i.e over all n) exponentials Z^n where Z is a complex number, are eigenfunctions (with $e^{j\omega n}$ a special case).

$\underbrace{H(z)}$ → provided it exists.

Proof

$$y(n) = h(n) * x(n) = h(n) * Z^n = \sum_{k=-\infty}^{\infty} h(k) Z^{n-k}$$

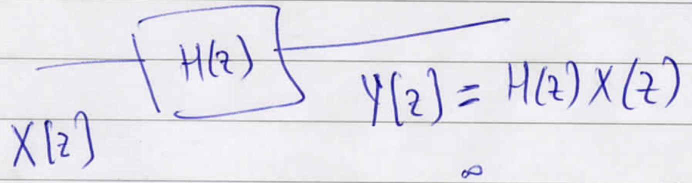
$$= Z^n \underbrace{\sum_{k=-\infty}^{\infty} h(z) Z^{-k}}_{H(z)} = Z^n H(z)$$

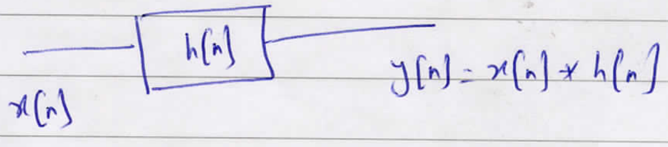
$H(z)$ ← if sum converges

→ Z Transform of $h(n)$

→ Using this property of Z^n we do (write everything, i.e., input & system as linear combo of Z^n)

this (freq. domain) | instead of this (time domain)





where $X[z] = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$ & $H[z] = \sum_{n=-\infty}^{\infty} h(n) Z^{-n}$

$X[z] = \mathcal{Z}\{x(n)\}$ $\quad$ $H[z] = \mathcal{Z}\{h(n)\}$

— and if we want output $Y(z)$ back in time-domain we can use

$$y(n) = Z^{-1}\{Y[z]\} = \frac{1}{2\pi j} \oint X[z]\, z^{n-1}\, dz$$

— This idea of "Frequency domain" has revolutionized engineering! (and been also useful in math & Physics)
→ solving Differential & Difference equations.

B

— So let's get to know the Z Transform better!

Defn. I $\quad X[z] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ $\quad$ bilateral

Defn. II $\quad X[z] = \sum_{n=0}^{\infty} x[n] z^{-n}$ $\quad$ unilateral

→ suitable for causal signals (that start at $n=0$)

C

— what is Z?

→ In general Z is a complex number
$Z = a + jb$ or $Z = Ae^{j\theta}$

→ and Z^n is an exponential

→ special case $Z = e^{j\omega}$, $Z^n = e^{j\omega n}$ (complex exponential) ← DT Fourier TX

D

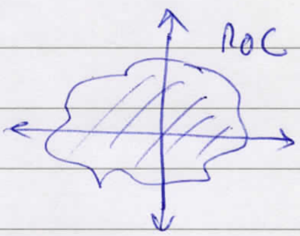
— ROC: Does $X[z]$ exist for any selection of Z?

where can you choose Z? Can you choose any Z?

— NO!

— Turns out for a given sequence $x[n]$ you can only choose values of Z for which the Z-Tx sum converges!

— So Z-Tx definition also includes a region of convergen (ROC) which may be different for different $x[n]$

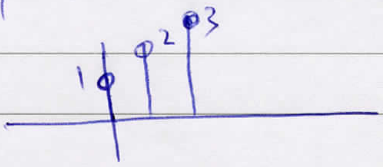


So — Defn. $X(Z) = \sum_{n=-\infty}^{\infty} x[n] Z^{-n}$

$\left.\begin{array}{l}\text{with } Z \text{ chosen such that the} \\ \text{sum converges!}\end{array}\right\}$ ROC

— Let's explore the ROC a bit further with examples.

— Let's say $x[n] = [\underset{\uparrow}{1}\ 2\ 3]$



→ then

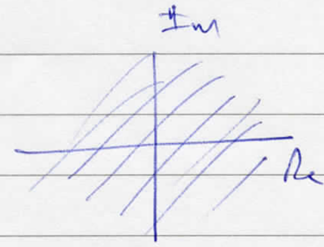
$$X(Z) = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1] z^{-1} + x[2] z^{-2}$$
$$= 1 + 2z^{-1} + 3z^{-2}$$

→ This sum will converge for any finite selection of complex number Z, so we say for this signal

$X[z] = 1 + 2z^{-1} + 3z^{-1}$ with ROC all $Z \neq 0$

ex2 → Try $x[n] = \delta[n]$

→ clearly $X[z] = 1$ with RoC all z



Notation → we write this as

$$\delta[n] \Longleftrightarrow 1 \quad \forall z$$

ex3 Try $x[n] = u[n]$

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

Q. when does this series converge?

$$X[z] = 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots = \sum ar^n \quad \begin{pmatrix} a = 1 \\ r = 1/z \end{pmatrix}$$

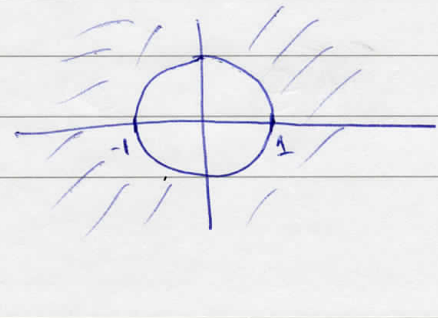
Power series

Q. when does a power series converge and what does it converge to?

$$\sum ar^n = \frac{a}{1-r} \quad \text{iff } |r| < 1$$

so in our case: $X[z] = \dfrac{1}{1 - \frac{1}{z}}$ iff $\left|\frac{1}{z}\right| < 1$

or, simply $X[z] = \dfrac{z}{z-1}$ with RoC $|z| > 1$



B

How to find Z-Tx — Matlab / Definition ← we did a few examples. / Properties + Tables

Tables in books

$$\delta[n] \Longleftrightarrow 1$$

$$\delta[n-k] \Longleftrightarrow z^{-k}$$

$$u(n) \Longleftrightarrow \frac{z}{z-1}$$

$$\gamma^n u(n) \Longleftrightarrow \frac{z}{z-\gamma}$$

ROC: often skipped in tables.

bilateral (unilateral mostly same, see book)

↓

Properties

1. Linearity ⇒ If $x_1(n) \Longleftrightarrow X_1[z]$ & $x_2[n] \Longleftrightarrow X_2[z]$

$$a_1 x_1[n] + a_2 x_2(n) \Longleftrightarrow a_1 X_1[z] + a_2 X_2[z]$$

new ROC = Intersection of $X_1[z]$ & $X_2[z]$ ROCs.

2. Shift

$$x[n-m] \Longleftrightarrow z^{-m} X[z]$$

see book (new ROC = {ROC of $X[z]$} plus or minus {$z=0$ or $z=\infty$}) ← depending on sign of m.

3. Convolution

$$x_1[n] * x_2[n] \Longleftrightarrow X_1[z]\, X_2[z]$$

new ROC = Intersection of $X_1[z]$, $X_2[z]$ ROCs

4. Multiplication by γ^n

$$\gamma^n x(n) \Longleftrightarrow X\left[\frac{z}{\gamma}\right]$$

ROC: see book.

5. Multiplication by n

$$n\, x[n] \Longleftrightarrow -z \frac{d X[z]}{dz}$$

6 Time reversal $x[-n] \Longleftrightarrow X[z^{-1}]$

7 complex Conjugate $x^*[n] \Longleftrightarrow X^*[z^*]$

F Practice

ex (i) $x(n) = (5 + 5^n) u(n)$, $X[z] = ?$

Soln: use linearity + Table 5.1

$$x[n] = \underbrace{5u[n]} + 5^n u[n]$$

Term 1 $u[n] \Longleftrightarrow \frac{z}{z-1}$ $\therefore$ $5u[n] \Longleftrightarrow \frac{5z}{z-1}$

Term 2 $\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma}$ $\therefore$ $5^n u[n] \Longleftrightarrow \frac{z}{z-5}$

$\rightarrow$ $x[n] \Longleftrightarrow \frac{5z}{z-1} + \frac{z}{z-5}$

ex (ii) $x[n] = 5^{n+1} u[n]$

by linearity

$\Rightarrow x(n) = 5\left(5^n u[n]\right)$

$5^n u[n] \Longleftrightarrow \frac{z}{z-5}$ $\therefore$ $5\left(5^n u[n]\right) \Longleftrightarrow \frac{5z}{z-5}$

ex iii) $x[n] = -2\delta[n-4] + \left[\frac{3}{2}(2)^n + \frac{5}{3}(3)^n\right] u[n]$ (with zero ICs)

Time shift + Linearity + Table.

$\Rightarrow X[z] = -2z^{-4} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$

$\delta[n] \leftrightarrow 1$, $\delta[n-4] \leftrightarrow z^{-4}$
$2^n u[n] \leftrightarrow \frac{z}{z-2}$
$3^n u[n] \leftrightarrow \frac{z}{z-3}$

Lec 14

(A) we have been familiarizing ourselves with the Z-Tx.

— Concept (need etc.)
— Definition
— Tables
— Properties
— How to find Z-Tx (some examples)

(B) we should also know how to find inverse Z-TX

— e.g. may need to get output $Y(Z)$ in time domain $y[n]$.

How to get Inv. Z-Tx ?
- (i) Matlab
- (ii) Partial fraction ~~Decomposition~~ Expansion + Properties + Table.
- (iii) Long division. (not closed form soln.) (reading exercise)

(C) Partial frac. ~~Dec.~~ (PF~~D~~E)

1. Apply PFE to $\frac{X[z]}{z}$
2. Multiply by Z
3. Use table

(why not $X[z]$? experience shows that $\frac{X[z]}{z}$ leads to better form, ie, in $u[n]$ rather than $u[n-1]$)

— usually

$$\frac{z}{z-\gamma} \Longleftrightarrow \gamma^n u[n]$$

~~(4)~~

ex 1 $$X[z] = \frac{8z-19}{(z-3)(z-2)}$$ ← distinct real roots.

step 1 $$\frac{X[z]}{z} = \frac{8z-19}{z(z-3)(z-2)} \triangleq F(z)$$

step 2 PFE: $$\frac{8z-19}{z(z-3)(z-2)} = \frac{a}{z} + \frac{b}{z-2} + \frac{c}{z-3}$$

$$a = z\,F(z)\Big|_{z=0}$$

$$b = (z-2)\,F(z)\Big|_{z=2}$$

$$c = (z-3)\,F(z)\Big|_{z=3}$$

Hint: see Background chapter $\boxed{B.5}$

e.g. $$c = \cancel{(z-3)}\left(\frac{8z-19}{z\cancel{(z-3)}(z-2)}\right)\Bigg|_{z=3}$$

$$= \frac{24-19}{3(1)} = \frac{5}{3}$$

Similarly: $a = -\frac{19}{6}$, $b = \frac{3}{2}$

$$\therefore \frac{X[z]}{z} = \frac{-19/6}{z} + \frac{3/2}{z-2} + \frac{5/3}{z-3}$$

Step ② Multiply by z

$$\Rightarrow X[z] = -19/6 + \frac{3}{2}\frac{z}{z-2} + \frac{5}{3}\frac{z}{z-3}$$

Step ③ use table

$$\delta[n] \Longleftrightarrow 1$$

$$\frac{z}{z-\cancel{\gamma}} \Longleftrightarrow \gamma^n u[n]$$

$$\Rightarrow x[n] = -\frac{19}{6}\delta[n] + \frac{3}{2}2^n u[n] + \frac{5}{3}3^n u[n]$$

ex2 Repeated Real Roots $X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

Step 1 $\frac{X[z]}{z} = \frac{2z^2-11z+12}{(z-1)(z-2)^3} \triangleq F(z)$

~~Step 2~~ PFE $F(z) = \frac{a}{z-1} + \frac{b}{z-2} + \frac{c}{(z-2)^2} + \frac{d}{(z-2)^3}$

$$\rightarrow a = (z-1)F(z)\Big|_{z=1}, \quad d = (z-2)^3 F(z)\Big|_{z=2}$$

$$\Rightarrow a = -3, \quad d = -2$$

(4)

$$\frac{2z^2-11z+12}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{b}{z-2} + \frac{d}{(z-2)^2}$$

Multiply both sides by z and set $z \to \infty$

$$0 = -3 - 0 + 0 + b \Rightarrow b = 3$$

e.g.

$$\lim_{z\to\infty} -3\frac{z}{z-1} = -3\times 1 = -3$$ (since z and $z-1$ approach infinity at the same rate)

$$\lim_{z\to\infty} -2\frac{z}{(z-2)^3} = -2\times 0 = 0$$ (since $(z-2)^3$ approaches ∞ ~~zero~~ much faster than numerator for z)

etc.

→ Finally get $d = -1$ by setting $z = 0$ (or any other convenient value)

Step ② Multiply by $z \Rightarrow X[z] = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{(z-2)}$

Step 3 Use table to get

$$x[n] = -3u[n] - 2\frac{n(n-1)}{8}2^n u(n) - \frac{n}{2}2^n u[n] + 3(2)^n u[n]$$

Example 3 complex root (Reading exercise)

Q. How to Model a system from history of inputs & outputs?

D claim 1: o/p of LTI Systems can also be modelled as linear combination of past inputs & outputs.

claim 2: Above approach for DT LTI systems leads to difference equations.

~~claim~~

claim 3: Z-TX helps represent & solve difference equations in an elegant way.

(converts difference equations into simple algebraic equations avoiding iterative solutions.)

→ Let us say you maintain a history of inputs & outputs

I/P	O/P
$x[n]$	$y[n]$
$x[n-1]$	$y[n-1]$
$x[n-2]$	$y[n-2]$
$\vdots$	$\vdots$
$x[n-N]$	$y[n-N]$

↓ past

→ for Linear Systems, current output $y[n]$ can be written as linear combos of these inputs & outputs

eg Marker on table: after three inputs, current state combined effect of three inputs.

but there can also be information in past outputs

eg if Marker pinned, then it gives rotary response.

$$y[n] = \overbrace{b_0 x[n] + b_1 x[n-1] + \dots + b_N x[n-N]}^{\text{Past inputs}} \underbrace{- a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N)}_{\text{Past outputs/states}} \quad \text{—} \; ⭐$$

↑ current output/state

Q. How to write & solve such $\boxed{\text{Difference eqns.}}$ efficiently?

$\Rightarrow$ Z-Tx

→ If we take Z-Tx of both sides, and use the fact (Property) that

$$x[n-1] \Longleftrightarrow z^{-1} X[z]$$

assumes zero initial conditions.

→ ⭐ reduces to

$$Y[z] = b_0 X[z] + b_1 z^{-1} X[z] + \dots + b_N z^{-N} X[z] - a_1 z^{-1} Y[z] - a_2 z^{-2} Y[z] \dots - a_N z^{-N} Y(z)$$

$$\Rightarrow Y[z] + a_1 z^{-1} Y[z] + \dots + a_N z^{-N} Y(z) = b_0 X[z] + b_1 z^{-1} X[z] + \dots + b_N z^{-N} X[z]$$

$$\Rightarrow \underbrace{\left[1 + a_1 z^{-1} + \dots + a_N z^{-N}\right]}_{Q(z)} Y[z] = \underbrace{\left[b_0 + b_1 z^{-1} + \dots + b_N z^{-N}\right]}_{P(z)} X[z]$$

$$\Rightarrow Y[z] = \frac{P(z)}{Q(z)} X[z] \Rightarrow H(z) = \frac{P(z)}{Q(z)}$$

→ let us look into some DT system models and related aspects.

ex

$$y[n] = 3x[n] + 0.8\,y[n-1]$$

solve for $x[n] = \delta[n]$ and $y[-1] = 10$ for $n \geq 0$

→ without using Z-TX we will have to solve iteratively

$$y[0] = 3\cancelto{1}{x[0]} + 0.8\cancelto{10}{y[-1]} = 3 + 8 = 11$$

$$y[1] = 3\cancelto{0}{x[1]} + 0.8\,y[0] = (0.8)(11)$$

$$y[2] = 3\cancelto{0}{x[2]} + 0.8\,y[1] = (0.8)(0.8)(11) = (0.8)^2(11)$$

$$\vdots$$

$$y[n] = \cdots\cdots = (0.8)^n(11) \rightarrow n \geq 0$$

$$= (0.8)^n(11)\,u(n)$$

→ may get event more complicated if $x[n]$ not so simple as $\delta[n]$!

$x[n] = \delta[n]$

$\delta[n] \Longleftrightarrow 1$

→ now let's try with Z-TX.

$$Y[z] = 3\cancelto{1}{X[z]} + 0.8\left(z^{-1}Y[z] + \cancelto{10}{y[-1]}\right)$$

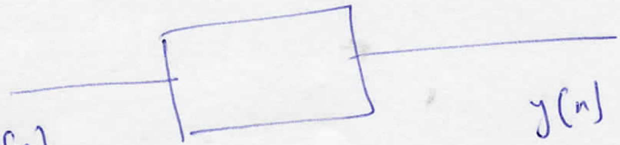
from delay property of unilateral Z-TX.

$$Y[z] = 3 + 8 + 0.8z^{-1}Y[z]$$

$$(1 - 0.8z^{-1})Y[z] = 11$$

$$Y[z] = \frac{11}{1 - 0.8z^{-1}} = \frac{(11)z}{z - 0.8} \Longleftrightarrow (11)(0.8)^n u(n) = y[n]$$

6) ZI vs ZS (separating effects of ICs and Input)



$x[n]$ $\quad$ $y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right)$

→ what's output for $x(n) = 0 \ \forall n$? (i.e find $y(0), y(1) \dots$)

→ $y[n] = 0 \quad \forall n$

→ what if we have

$$y[n] = 3x[n] + 0.8\, y[n-1] \ ?$$

and $x[n] = 0 \ \forall n$?

→ is $y[n]$ still zero?

→ depends on initial condition!

→ let's say $y[-1] = 10$

→ then

$$y[0] = 0.8(10) = 8$$
$$y[1] = 0.8\, y[0] = 8(0.8)$$
$$y[2] = 0.8\, y[1] = 8(0.8)^2$$
$$\vdots$$
$$y[n] = \quad = 8(0.8)^n \quad n \geq 0$$

ZIR

→ what if Initial condition/state zero and then we give input, e.g $x[n] = \delta[n]$, and $y[-1] = 0$

$$\rightarrow y[0] = 3 + 0.8\, \cancel{y[-1]}^{0} = 3$$

$$y[1] = 3\cancel{x[1]}_{0} + 0.8\, y[0] = 3(0.8)$$

$$y[2] = 3\cancel{x[2]}_{0} + 0.8\, y[1] = 3(0.8)^2$$

$$\vdots$$

$$y(n) = \ldots = 3(0.8)^n \quad n \geq 0$$

ZSR

→ what if state not zero and input $\delta(n)$?

→ Total Repone !

= ZIR + ZSR.

$$y[n] = 8(0.8)^n + 3(0.8)^n \quad n \geq 0$$

$$= 11(0.8)^n u[n]$$

Lec15 Practice

Ex1 DT LTI

$$x[n] = \begin{cases} 1 & 1 \le n \le 6 \\ 0 & o/w \end{cases}$$

$$h[n] = \begin{cases} 2 & -2 \le n \le 1 \\ 0 & o/w \end{cases}$$

find $y[n]$.

soln $y[n] = x[n] * h[n]$

→ one way: Tape method

0	1	1	1	1	1	1	0	$x[n]$

2	2	2	2

1	1	1	1	1	1	0

$y[0] = 2\times1 + 2\times1 = 4$

2	2	2	2

1	1	1	1	1	1	0

$y[1] = 6$

etc.

→ second way: use defn. of convolution.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k] = \underbrace{x[-1]h[1]}_{0} + \underbrace{x[0]h[0]}_{0} + \underbrace{x[1]h[-1]}_{1\times2} + \underbrace{x[2]h[-2]}_{2\times1} + \underbrace{x[3]h[-3]}_{0} + \underbrace{x[-2]h[2]}_{0}$$

$= 4$

etc.

②

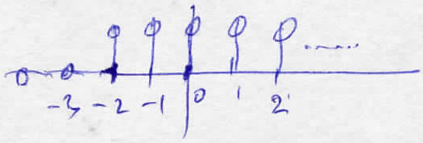
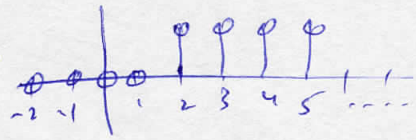
DT LTI

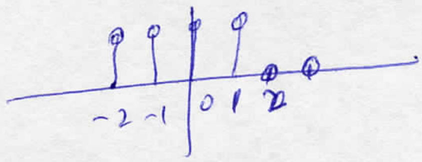
EX 2 Given $h[n] = u[n+2] - u[n-2]$

$x[n] = (0.7)^n u[n]$

$\boxed{IC_s = 0}$

ⓐ sketch $h[n]$

 $-$ 

$h[n] =$ 

ⓑ Find output $y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-2}^{1} h[k]\, x[n-k]$$

$$= \sum_{k=-2}^{1} (0.7)^{n-k} u[n-k]$$

$$= (0.7)^{n+2} u[n+2] + (0.7)^{n+1} u[n+1] + (0.7)^{n} u[n] + (0.7)^{n-1} u[n-1]$$

for $n \le -3$: all terms vanish $= 0$

$n = -2$: $(0.7)^0 = 1$

$n = -1$: $(0.7)^1 + (0.7)^0 = 1.7$

$n = 0$: $(0.7)^2 + (0.7)^1 + (0.7)^0 = 2.19$

$n = 1$: $(0.7)^3 + (0.7)^2 + (0.7)^1 + (0.7)^0$

$n = 2$: $(0.7)^4 + (0.7)^3 + (0.7)^2 + (0.7)^1$

$n \ge 1$: $(0.7)^{n+2} + (0.7)^{n+1} + (0.7)^{n} + (0.7)^{n-1}$

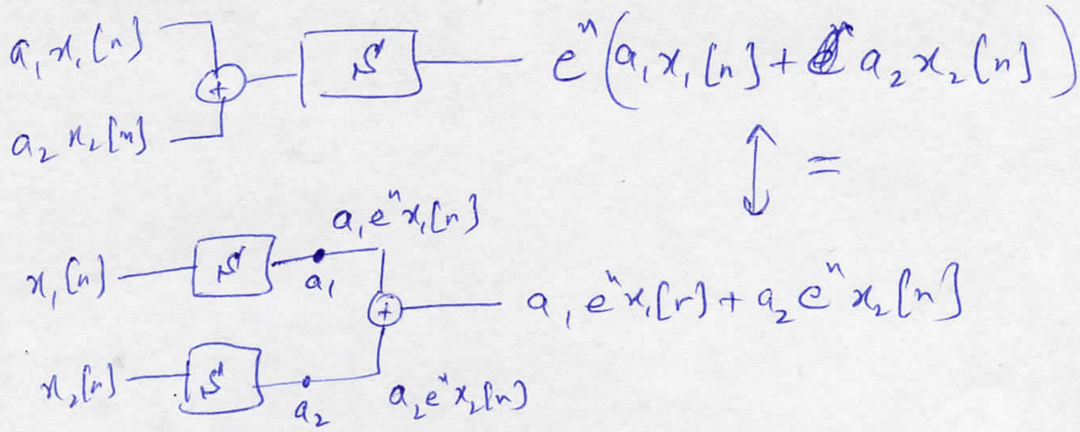
ⓒ stable? Yes $\sum |h[n]| < \infty$

ⓓ causal? No $h[n] \neq 0 \quad n < 0$

ⓔ Memoryless? no! $h[n] \neq 0 \quad n \neq 0$

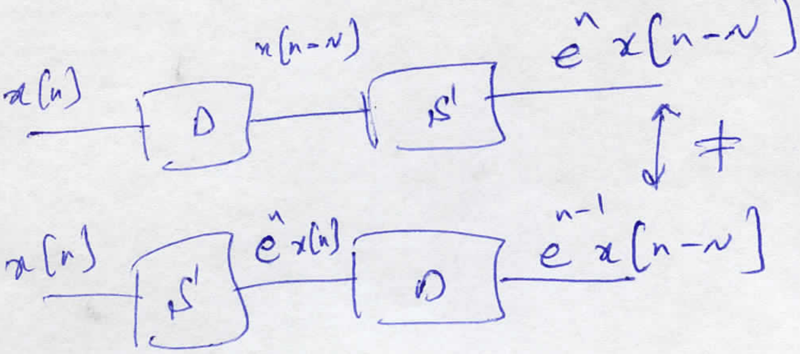
ex3 consider $y[n] = e^n x[n] \Rightarrow S[x[n]] = e^n x[n]$

(a) linear?



→ linear ✓

(b)



shortcut: if (n) appears free of $x[n]$ then NOT TI

→ TI ✗

(c) Find Impulse response $h[n]$

→ when $x[n] = \delta[n]$ we have $y[n] = h[n]$

$$\Rightarrow \quad h[n] = e^n \delta[n] = e^0 \delta[n] = \delta[n]$$

(d) Find output to $\delta[n-1]$ with and without convolution. same result? why?

A → without convolution

$$y[n] = \mathcal{S}\{x[n]\} = e^{n} x[n]$$

$$\Rightarrow y[n] = \mathcal{S}\{\delta[n-1]\} = e^{n}\delta[n-1] = e^{1}\delta[n-1] = e\,\delta[n-1]$$

B → with convolution

$$y[n] = h[n] * \delta[n-1]$$

$$= \delta[n] * \delta[n-1]$$

$$= \delta[n-1]$$

since $\delta[n]$ is identity of convolution

→ Results A ≠ B

→ Second one is wrong

→ why? because convolution only works for LTI!

Q) A Time invariant system

$x_1[n] = 2\delta[n-3]$ → $y_1[n] = 2\delta[n-2] + 4\delta[n-3]$

$x_2[n] = 2\delta[n-1] + 2\delta[n-3]$ → $y_2[n] = 2\delta[n-2] + 6\delta[n-3] + 4\delta[n-4]$

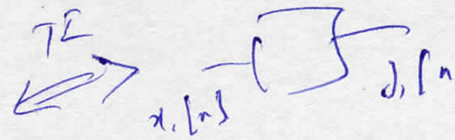
Prove that the system is nonlinear. (5)

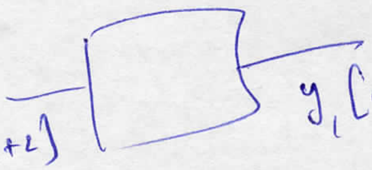
Soln we first note that

$$x_2[n] = x_1[n+2] + x_1[n]$$

Since system is TI, output to $x_1[n+2]$

should be





$x_3[n] \triangleq x_1[n+2]$ $\quad$ $y_1[n+2]$

$$= 2\delta[n] + 4\delta[n-1] \triangleq y_3[n]$$

ie

→ second output to $x_1[n] + x_1[n+2]$ should equal sum of outputs to each individually, i.e

$$y_{LTI}[n] = y_1[n] + y_3[n]$$

$$y_{LTI} = 2\delta[n-2] + 4\delta[n-3] + 2\delta[n] + 4\delta[n-1]$$

but given $y_2[n]$ is different. So system not LTI!

$\underline{EX}$ find the even & odd components of

$$x(t) = e^{jt}$$

$\underline{\text{w.k.t}}$
$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$
$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

$$\Rightarrow \left.\begin{aligned} x_e(t) &= \frac{1}{2}\left(e^{jt} + e^{-jt}\right) = \cos t \\ x_o(t) &= \frac{1}{2}\left(e^{jt} - e^{-jt}\right) = j\sin t \end{aligned}\right\} \text{by Euler}$$

$\underline{\text{or}}$
$$e^{jt} = \underbrace{\cos t}_{\text{even}} + \underbrace{j\sin t}_{\text{odd}}$$

$\underline{EX}$ Let $x_1(t)$ and $x_2(t)$ be periodic with fund. periods T_1 and T_2. Under what condition is the sum $x(t) = x_1(t) + x_2(t)$ periodic?

$\underline{\text{Sln}}$ $x(t)$ is periodic with period T if

$$x(t+T) = x(t)$$

$$\Rightarrow x_1(t+T) + x_2(t+T) = \overbrace{x_1(t) + x_2(t)}^{\text{RHS}}$$

can we get some clue about T?

we know that $x_1(t) + x_2(t) = x_1(t+T_1) + x_2(t+T_2)$

$$= x_1(t+mT_1) + x_2(t+kT_2)$$

for periodicity
→ So, T must be simultaneously equal to mT_1 and kT_2

$$\Rightarrow \underbrace{mT_1 = kT_2}_{} = T \quad \text{—} \quad ☆$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{m}{k} = \text{Rational number.}$$

Integer
Integer

→ so $\frac{T_1}{T_2}$ must lead to a rational number for sum to be periodic

→ $T = LCM(T_1, T_2)$ — ☆ indicates that they have a common multiple.

So Sum of two periodic ~~fun~~ functions is periodic iff their periods ~~have~~ form a rational number and the period of the new function is the LCM of the periods.

→ e.g $T_1 = 10$, $T_2 = 5$

$$\frac{T_1}{T_2} = \frac{10}{5} = 2, \quad T = LCM(T_1, T_2) = 10$$

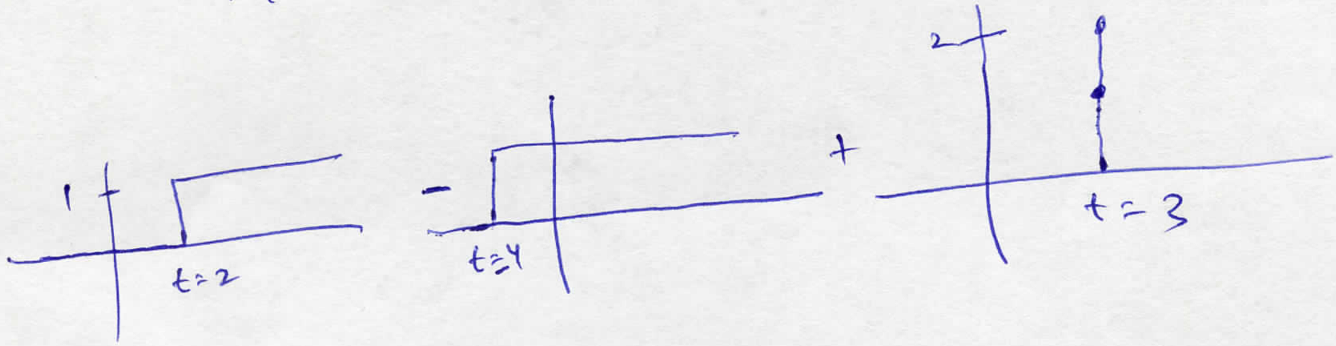
but

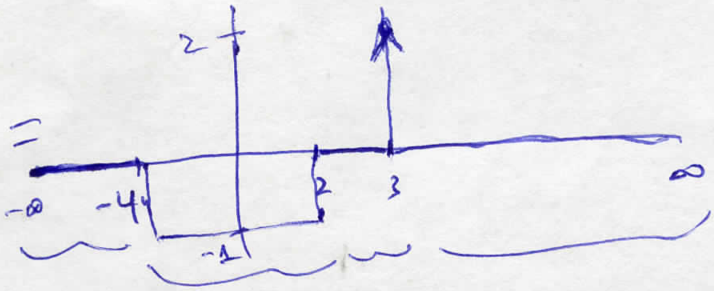
$\boxed{\begin{array}{l} \text{if } T_1 = \pi,\ T_2 = 1 \\ \frac{T_1}{T_2} = \pi \neq \text{rational} \\ LCM(T_1, T_2) \text{ does not exist.} \end{array}}$

ex

Plotting

$$x(t) = u(t-2) - u(t+4) + 2\delta(t-3)$$





$$\int_{-\infty}^{\infty} x(t)\,dt = -6 + 2 = 4$$

Lec 16

→ we now repeat a lot of what we did ~~for~~ in DT for CT

→ Nearly all concepts, definitions etc. remain same!

DT → CT
- → $n \rightarrow t$ [time domain]
- → $z^n \rightarrow e^{st}$ [$s = \sigma + j\omega$] [freq. domain]
- → $\Sigma \rightarrow \int$ [operations].

Systems → Definition & ways of checking LTI remain same.

Signals → $\boxed{\text{IMPULSE}}$

DT	CT
– $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & o/w \end{cases}$	– $\delta(t)$ s.t. $\delta(t) = 0 \quad t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$
– ~~$\delta[N]$~~ $\delta[n-N]x[n] = \delta[n-N]x[N]$	– $\delta(t-T)\phi(t) = \delta(t-T)\phi(T)$
– $\sum_{\forall n} x[n]\delta[n-N] = x[N]$	– $\int_{-\infty}^{\infty} \phi(t)\delta(t-T)\,dt = \phi(T)$
– $x[n] * \delta[n] = x[n]$	– $x(t) * \delta(t) = x(t)$
↓ – $x[n] = \sum_{\forall k} x[k]\delta[n-k]$ (sum of scaled & shifted impulses.)	↓ – $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$

→ LTI IR

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$$

$$\cancel{a_0\delta(t)} \longrightarrow \boxed{\;} \longrightarrow \cancel{a_0 h(t)}$$

$$\cancel{a_1\delta(t-T)} \longrightarrow \boxed{\;} \longrightarrow \cancel{a_1 h(t-T)}$$

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}_{x(t)} \longrightarrow \boxed{\;} \longrightarrow \underbrace{\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}_{y(t)}$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \triangleq x(t) * h(t)$$

→ conv. properties similar to DT case (see Lathi 2.4.1 & 2.4.2)

→ characterization of system via IR
<u>also</u> similar to DT case.

- e.g. causal iff $h(t) = 0 \quad t < 0$
- BIBO stable iff $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$
- etc.

→ Freqeuncy Domain

$\gamma^n \longrightarrow \boxed{\text{DT-LTI}} \longrightarrow \alpha\gamma^n$ eigenfunctions (γ complex)

$e^{st} \longrightarrow \boxed{\text{CT-LTI}} \longrightarrow \alpha e^{st}$ (s complex)

→ $s = \sigma + j\omega \Rightarrow e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t} e^{j\omega t}$

→ Real and complex parts fixed/damped/growing sinusoids.

→ $Z^{Tx} \longrightarrow$ Laplace Tx.

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$ Bilateral Z-Tx

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$ Bilateral Laplace-Tx.

→ Again, what is s? $s = \sigma + j\omega$ $e^{st} = e^{\sigma t} e^{j\omega t}$

→ ROC: Does $X(s)$ exist for any selection of ~~σ~~ s?

→ for a given signal $x(t)$ we can only choose values of s for which ✪ converges!

[detailed practice problem later]

→ properties & Tx-Tables

→ similar to Z-Tx.

→ Unilateral Laplace Tx.

$$X(s) = \int_0^\infty x(t) e^{-st} dt$$

(avoids some issues of inverse Tx faced by bilateral)

→ Laplace Tx examples.

– $$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt$$

$$= e^{-st}\Big|_{t=0} \int_0^\infty \delta(t)\, dt = 1$$

$\boxed{\text{ROC} \quad \text{converges for all } s}$

$$\delta(t) \Longleftrightarrow 1$$

for $e^{-(s+a)t}$ to vanish → $\mathrm{Re}(s+a) > 0$

– $$\mathcal{L}[u(t)] = \int_0^\infty u(t) e^{-st} dt = \int_0^\infty e^{-st} dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$$

$\boxed{\text{ROC} \quad \mathrm{Re}\, s > 0}$

– $$\mathcal{L}[e^{-at} u(t)] = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt$$

$\boxed{\text{ROC} \quad \mathrm{Re}\, s > -a}$

$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}$$

→ Some properties.

→ Linear. if $x_1(t) \Leftrightarrow X_1(s)$
$x_2(t) \Leftrightarrow X_2(s)$

then $a_1 x_1(t) + a_2 x_2(t) \Leftrightarrow a_1 X_1(s) + a_2 X_2(s)$

→ Time shifting if $x(t) \Leftrightarrow X(s)$

then $x(t-t_0) \Leftrightarrow X(s)e^{-st_0}$

→ Freq. shifting.

if $x(t) \Leftrightarrow X(s)$

then $x(t)e^{s_0 t} \Leftrightarrow X(s-s_0)$

→ Differentiation (time) property

if $x(t) \Leftrightarrow X(s)$

initial condition..

then $\dfrac{dx(t)}{dt} \Leftrightarrow sX(s) - x(0)$

and $\dfrac{d^2x(t)}{dt^2} \Leftrightarrow s^2X(s) - \underbrace{sx(0) - \dot{x}(0)}$

etc.

→ Integration (Time) property

if $x(t) \Leftrightarrow X(s)$

then $\int_0^t x(\tau)\,d\tau \Leftrightarrow \frac{X(s)}{s}$

→ Scaling (time)

if $x(t) \Leftrightarrow X(s)$

then $x(at) \Leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$

→ Convolution (Time & freq).

if $x_1(t) \Leftrightarrow X_1(s)$
$x_2(t) \Leftrightarrow X_2(s)$

then $x_1(t) * x_2(t) \Leftrightarrow X_1(s)\,X_2(s)$

and $x_1(t)\,x_2(t) \Leftrightarrow \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$

→ Initial & final value of $x(t)$

$x(0^+) = \lim_{s\to\infty} sX(s)$

$x(\infty) = \lim_{s\to 0} sX(s)$

Lec 17

→ we saw that DT-LTI systems could be modeled as Difference equations (of I/P & O/P)

→ and that these difference equations could be converted into simple algebraic (non-iterative) equations using Z-TX

→ particularly, using

$$x[n-k] \Leftrightarrow z^{-k} X(s) \quad \text{for zero ICs.}$$

→ e.g a DT LTI system:

$$y[n] = 3x[n] + 0.8\, y[n-1]$$

→ Described by difference eqn. (contains delay samples etc.).

→ Using Z with assumption ICs = 0 (for simplicity)

$$\Rightarrow \quad Y[z] = 3X[z] + 0.8 z^{-1} Y[z]$$

$$\Rightarrow \quad Y[z] = \frac{3X[z]}{1 - 0.8z^{-1}}$$

$$Y[z] = H[z] X[z]$$

→ simple algebraic eqn. in Z-domain.

→ on similar lines

1. CT-LTI systems are represented by differential equations!

2. And using Laplace Tx. these differential equations can be converted into simple algebraic equations.

3. we will make use of the fact that

$$\frac{d^k y(t)}{dt^k} \Longleftrightarrow s^k Y(s)$$ Assuming all ICs = 0.

→ e.g. $$5\frac{dy(t)}{dt} + 6y(t) = 4x(t)$$ ICs = 0

Differential eqn. →

$$\Rightarrow \quad 5sY(s) + 6Y(s) = 4X(s)$$

$$(5s+6)Y(s) = 4X(s)$$

$$Y(s) = \frac{4}{5s+6}X(s)$$

$$Y(s) = H(s)X(s)$$

→ simple algebraic eqn.

→ In general, an CT-LTI system described by N-th order DE (assuming zero ICs for convenience).

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1}\frac{dy(t)}{dt}$$

$$= b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \dots + b_N x(t)$$

→ This can be simplified using Laplace as

$$s^N Y(s) + a_1 s^{N-1} Y(s) + \dots + a_{N-1} Y(s) = b_0 s^N X(s) + b_1 s^{N-1} X(s) + \dots + b_N X(s)$$

$$\Rightarrow Y(s)\left(s^N + a_1 s^{N-1} + \dots + a_{N-1}\right) = X(s)\left(b_0 s^N + b_1 s^{N-1} + \dots + b_N\right)$$

$$Y(s) = \frac{b_0 s^N + b_1 s^{N-1} + \dots + b_N}{s^N + a_1 s^{N-1} + \dots + a_{N-1}} X(s)$$

$$Y(s) = H(s)\, X(s)$$

→ $H(s)$ = System Transfer function.

→ Also Zero-state Response (since we assumed ICs = 0).

→ Checking stability of CT-LTI from $H(s)$

→ BIBO stable from $h(t)$ if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

→ what about BIBO stability from $H(s)$?

→ of Zeros & Poles

Zeros: values of s that make $H(s) = 0$

Poles values of s that make $H(s) = \infty$

→ For rational case (i.e LTI case) $H(s)$ can be written as rational function ~~eg~~

$$H(s) = \frac{P(s)}{Q(s)} = \frac{(s-b_1)(s-b_2)\cdots(s-b_N)}{(s-a_1)(s-a_2)\cdots(s-a_N)}$$

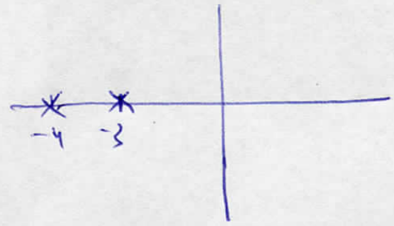
$\Rightarrow H(s) = 0$ for $s = b_1, b_2, \ldots, b_N$

$H(s) = \infty$ for $s = a_1, a_2, \ldots, a_N$

→ Case: all poles in left half plane

$$H(s) = \frac{1}{s+3} + \frac{2}{s+4} \Rightarrow h(t) = \underbrace{e^{-3t}u(t) + 2e^{-4t}u(t)}_{\text{both decaying functions.}}$$

$$\Rightarrow \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$



if all poles in left half plane (as they lead to decaying functions in time domain)

→ case: even a single pole in RHP

$$H(s) = \frac{1}{s+3} + \frac{2}{s-4}$$

$$\Rightarrow h(t) = \underbrace{e^{-3t}u(t)}_{\text{decaying}} + 2\underbrace{e^{4t}u(t)}_{\text{exponentially growing.}}$$

Now

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \infty.$$

$\therefore$ CT-LTI BIBO stable iff all poles of $H(s)$ in LHP!

↓

Assuming $H(s)$ in reduced form (common pole and zeros cancelled)

→ Responses
1. Zero-State Response : → Give input → Assume ICs = 0
2. Zero-Input Response : → No input → ICs ≠ 0
3. Total Response = ZSR + ZIR : → Give input → ICs ≠ 0

→ example: Consider CT-LTI system

$$\frac{d^2 y(t)}{dt^2} + \frac{5y(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

1. → let's do $\underline{\underline{ZSR}}$ i.e $IC_s = 0$

→ some input! let's say $x(t) = e^{-4t} u(t)$
$\left(\Rightarrow X(s) = \frac{1}{s+4} \text{ table} \right)$

→ Find $y(t)$!

→ note: easier if we use Laplace

→ using Laplace table with zero IC_s gives

$$s^2 Y(s) + 5sY(s) + 6Y(s) = sX(s) + X(s)$$

$$\Rightarrow Y(s) = \frac{s+1}{s^2+5s+6} X(s) = \frac{s+1}{(s^2+5s+6)(s+4)}$$

→ using PFE $\quad Y(s) = \frac{-\frac{1}{2}}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}$

→ From table of $\mathcal{L}$

$$y(t) = \left(-\frac{1}{2} e^{-2t} + 2e^{-3t} - \frac{3}{2} e^{-4t}\right) u(t)$$

→ $\boxed{ZSR}$

(2) → Now let's do Zero-Input Response.

→ $x(t) = 0 \quad \forall t$

→ some IC_s! e.g. $y(0) = 2$, $\dot{y}(0) = 1$

→ same system, but now

$$\frac{dy(t)}{dt} \Longleftrightarrow sY(s) - y(0)$$

etc.

no I/P

$$\Rightarrow \underbrace{\left[s^2 Y(s) - 2s - 1\right]}_{\frac{d^2y(t)}{dt}} + 5\underbrace{\left[sY(s) - 2\right]}_{\frac{dy(t)}{dt}} + \underbrace{6Y(s)}_{y(t)} = 0$$

from IC_s (not I/P)

$$\Rightarrow s^2 Y(s) + 5sY(s) + 6Y(s) = 2s + 11$$

$$\Rightarrow Y(s) = \frac{2s+11}{s^2+5s+6} = \frac{7}{s+2} - \frac{5}{s+3}$$

→ using Laplace Table: $y(t) = \left(7e^{-2t} - 5e^{-3t}\right)u(t)$

$\boxed{ZIR}$

(3) The case of Total Response i.e.

→ ICs ≠ 0

→ Input ≠ 0

→ for example above, you can verify that

$$TR = ZIR + ZSR$$

$$\Rightarrow y(t) = \left(7e^{-2t} - 5e^{-3t}\right)u(t)$$

$$+ \left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t)$$

Lec 18 Practice (Laplace)

Ex 1 $X(s) = \dfrac{7s-6}{s^2-s-6}$ find $x(t)$.

$-3 \times 2 = -6$
$-3 + 2 = -1$

$$X(s) = \frac{7s-6}{s^2-3s+2s-6}$$

$$= \frac{7s-6}{s(s-3)+2(s-3)} = \frac{7s-6}{(s+2)(s-3)} = \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

cover up method:

$$k_1 = \left.\frac{7s-6}{\boxed{\cancel{(s+2)}}(s-3)}\right|_{s=-2} = \frac{-14-6}{-2-3} = 4$$

covered-up

similarly

$$k_2 = \left.\frac{7s-6}{(s+2)\boxed{\cancel{(s-3)}}}\right|_{s=3} = \frac{21-6}{3+2} = 3$$

$$X(s) = \frac{4}{s+2} + \frac{3}{s-3}$$

Quick check put $s=0$ (or any convenient value) to see original & expanded versions give same answer (note: though this does not guarantee correctness, it gives some confidence)

→ use Laplace table $\frac{1}{s+a} \Leftrightarrow e^{-at}u(t)$

$$\Rightarrow x(t) = \left(4e^{-2t} + 3e^{3t}\right)u(t)$$

Ex.2 $X(s) = \frac{2s^2+5}{s^2+3s+2}$, find $x(t)$.

→ This is M=N case (same ~~highe~~ order in num and den)

→ shortcut for such cases:
separate out coefficient of highest power.

$\boxed{\begin{matrix} 2\times 1 = 2 \\ 2+1 = 3 \end{matrix}}$

$$\Rightarrow X(s) = \frac{2s^2+5}{s^2+3s+2} = \frac{2s^2+5}{(s+1)(s+2)}$$

$$= 2 + \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

→ now apply cover-up rules, to get.

$$K_1 = 7 \quad \text{and} \quad K_2 = -13$$

$$\Rightarrow X(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2}$$

→ Use Laplace table $\left\langle \begin{matrix} \frac{1}{s+a} \Leftrightarrow e^{-at}u(t) \\ 1 \Leftrightarrow \delta(t) \end{matrix} \right.$

$$\Rightarrow x(t) = 2\delta(t) + \left(7e^{-t} - 13e^{-2t}\right)u(t)$$

Ex3 $X(s) = \dfrac{5e^{-2s}}{(s+1)(s+2)}$ find $x(t)$.

→ we note that e^{-2s} represents delayed version.

e.g. if $x_1(t) \Longleftrightarrow X_1(s)$

then $x_1(t-2) \Longleftrightarrow e^{-2s} X_1(s)$

$$\Rightarrow X(s) = \underbrace{\left(\frac{5}{(s+1)(s+2)}\right)}_{X_1(s)}\left(e^{-2s}\right)$$

$$\Rightarrow X_1(s) = \frac{5}{(s+1)(s+2)} = \frac{5}{s+1} - \frac{5}{s+2} \quad \text{(PFE)}$$

$$\Rightarrow x_1(t) = \left(5e^{-t} - 5e^{-2t}\right)u(t)$$

$$\therefore x(t) = x_1(t-2) = \left(5e^{-(t-2)} - e^{-2(t-2)}\right)u(t-2)$$

Ex4 Given $e^{-at}\cos bt\, u(t) \Longleftrightarrow \dfrac{s+a}{(s+a)^2+b^2}$

Find $\mathcal{L}\{\cos bt\, u(t)\}$.

→ set $a = 0$

$$\cos bt\, u(t) \Longleftrightarrow \frac{s}{s^2+b^2}$$

Ex 5 Assuming zero ICs, Find ~~solve DE~~ via $\mathcal{L}$-Tx.

$$\frac{d^2x(t)}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$

$\Rightarrow$ Taking $\mathcal{L}$-Tx

$$s^2X(s) - s\cancel{x(0)}^{0} - \cancel{\dot{x}(0)}^{0} = 1 - 3e^{-2s} + 2e^{-3s}$$

$$\Rightarrow \quad X(s) = \frac{1}{s^2}\left(1 - 3e^{-2s} + 2e^{-3s}\right)$$

Ex 6 use $\mathcal{L}$-Tx to find the conv.

$$c(t) = \underbrace{e^{at}u(t)}_{x_1(t)} * \underbrace{e^{bt}u(t)}_{x_2(t)}$$

$$\rightarrow \quad x_1(t) \Longleftrightarrow \frac{1}{s-a}$$

$$x_2(t) \Longleftrightarrow \frac{1}{s-b}$$

Taking $\mathcal{L}$-Tx

$$\rightarrow \quad C(s) = X_1(s)\,X_2(s) = \frac{1}{(s-a)(s-b)}$$

using PFE $\rightarrow$

$$C(s) = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)$$

$$\Rightarrow c(t) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)u(t)$$

Slides Mostly — Lec 19 + Lec 20

1. We saw that Fourier Tx

= Given a time domain signal
Find which complex sinusoids need to be combined to form this signal.

→ FT records the amplitudes & phases of the required complex sinusoids.

2. Defn.

→ FT: write $x(t)$ as a combination of complex sinusoids ($e^{j\omega t}$).

2π appears b/c of $\omega = 2\pi f$

$$\Rightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad \text{IFT}$$

$$\Rightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{FT}$$

3. → $X(\omega)$ is a complex variable.

$$X(\omega) = Re(X(\omega)) + j\,Im(X(\omega))$$

$$X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)}$$

→ stores information on magnitudes & phases of complex sinusoids required to form the signal.

(4) Let's get to know the mathematics of FT !!

(a) What is $\int_{-\infty}^{\infty} e^{-j\omega t}\, dt = ?$ ——— (★)

⇒ There are two cases:

for $\omega \neq 0$ $\quad e^{-j\omega t}$ is a periodic function

$$e^{-j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

for $\omega = 0$ $\quad e^{-j\omega t} = e^{0} = 1$ is a constant.

$$e^{-j(0)t} = \cos(0) + j\sin(0) = 1.$$

⇒ You can convince yourselves that for $\omega \neq 0$, the total area under $e^{-j\omega t}$ equals zero. (e.g. by drawing real & complex parts)

⇒ for $\omega = 0$ we have

$$\int_{-\infty}^{\infty} e^{0}\, dt = \int_{-\infty}^{\infty} 1 . dt = t \Big|_{-\infty}^{\infty} = \infty$$

⇒ so (★) is zero for all values of ω, except $\omega = 0$ where it has infinite amplitude.

⟶ we know such a function : Continuous-time Impulse $\delta(\omega)$

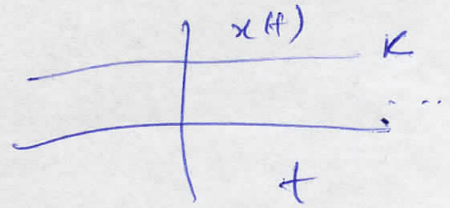
$$\therefore \int_{-\infty}^{\infty} e^{-j\omega t}\, dt = 2\pi\, \delta(\omega)$$

→ the 2π appears due to $\omega = 2\pi f$

→ if we had $\int_{-\infty}^{\infty} e^{-jft}\, dt$ result would be just $\boxed{\delta(\omega)}$

(b) → using the above, let's find FT of a constant (k)

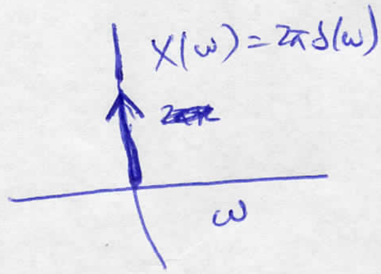
$x(t) = k \quad \forall t$



$$\text{FT:} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} k\, e^{-j\omega t}\, dt$$

$$= k \int_{-\infty}^{\infty} e^{-j\omega t}\, dt$$ → same as (*)

$$= k\left(2\pi\, \delta(\omega)\right) = 2\pi k\, \delta(\omega) \Rightarrow$$

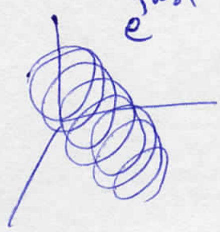


→ this makes sense, since a constant has zero frequency (resulting in $X(\omega)$ having a peak at $\omega = 0$, and zero everywhere else)

(4)

(c) Now, let's find FT of a single complex sinusoid

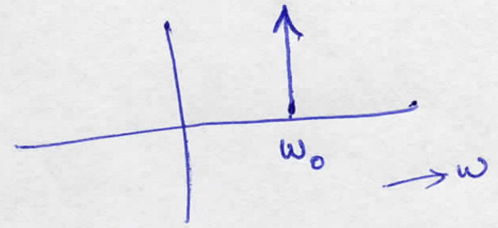
e.g. $x(t) = e^{jw_0 t}$ where w_0 = constant.



$$X(w) = \int_{-\infty}^{\infty} e^{jw_0 t} e^{-jwt} \, dt = \int_{-\infty}^{\infty} e^{j(w_0 - w)t} \, dt$$

→ which is similar to (A) except that now $e^{j(w_0 - w)t}$ is a constant at $w = w_0$

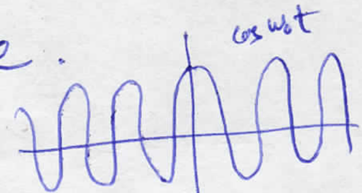
→ This gives a shifted Impulse!



$$X(w) = 2\pi\delta(w - w_0)$$

→ which makes sense, since it sayst that to form $e^{jw_0 t}$ we just need the complex sinusoid of frequency w_0.

(d) Now let's find FT of a pure cosine.



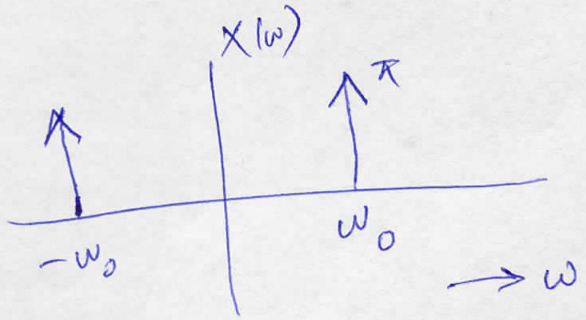
$x(t) = \cos(w_0 t)$ $\quad w_0$ = constant.

→ Using Euler $\cos(w_0 t) = \frac{1}{2} e^{jw_0 t} + \frac{1}{2} e^{-jw_0 t}$ —— (Δ)

$$\rightarrow X(s) = \int_{-\infty}^{\infty} \cos(w_0 t) \, dt = \frac{1}{2} \int_{-\infty}^{\infty} e^{jw_0 t} \, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-jw_0 t} \, dt$$

using part (c) $\rightarrow X(s) = \frac{1}{2}\left(2\pi\delta(w + w_0) + 2\pi\delta(w - w_0)\right) = \pi\left(\delta(w + w_0) + \delta(w - w_0)\right)$

→ which make sense since (A) also tells us that to form $\cos(w_0 t)$ you need two complex sinusoids, one at w_0 and one at $-w_0$



(C) Let's find FT of an exponential (not complex)

$$x(t) = e^{-at} u(t) \qquad a: \text{Real}$$

→ There are two cases

case (1) $a > 0$

$$X(w) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-jwt} \, dt = \int_{0}^{\infty} e^{-at} e^{-jwt} \, dt$$

$$= -\frac{1}{a+jw} e^{-at} e^{-jwt} \Big|_0^{\infty}$$

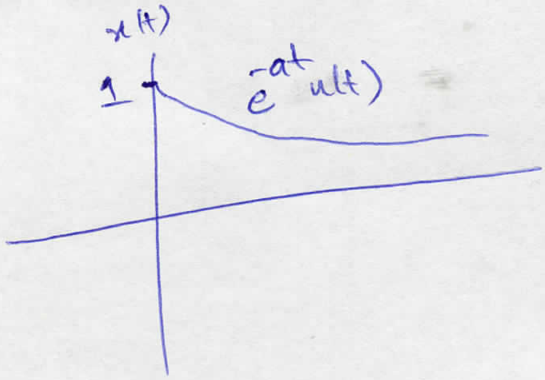
→ oscillating function with modulus always ~~less than~~ equal to 1.

→ exponentially decaying since $a > 0$ and $t \geq 0$

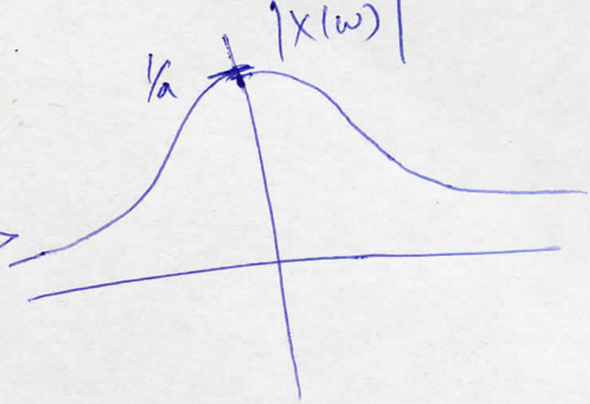
$$\Rightarrow X(w) = \frac{1}{a+jw} \qquad a > 0$$

$$\Rightarrow |X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$a>0$



$\Longleftrightarrow$



Case (ii) $a<0$

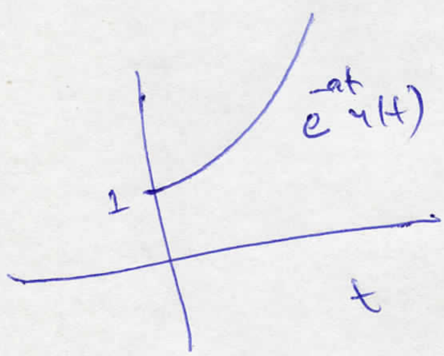
$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= -\frac{1}{a+j\omega} \left. e^{-at} e^{-j\omega t} \right|_{0}^{\infty}$$

$\rightarrow$ exponentially growing function since $a<0$ and $t \geq 0$

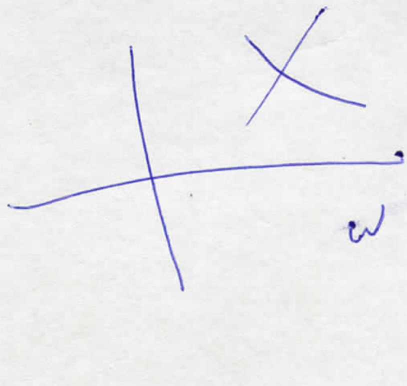
$$\Rightarrow X(\omega) = \infty = \boxed{\text{FT Does Not Exist}}$$

$a<0$



$\Longleftrightarrow$

NO FT!!



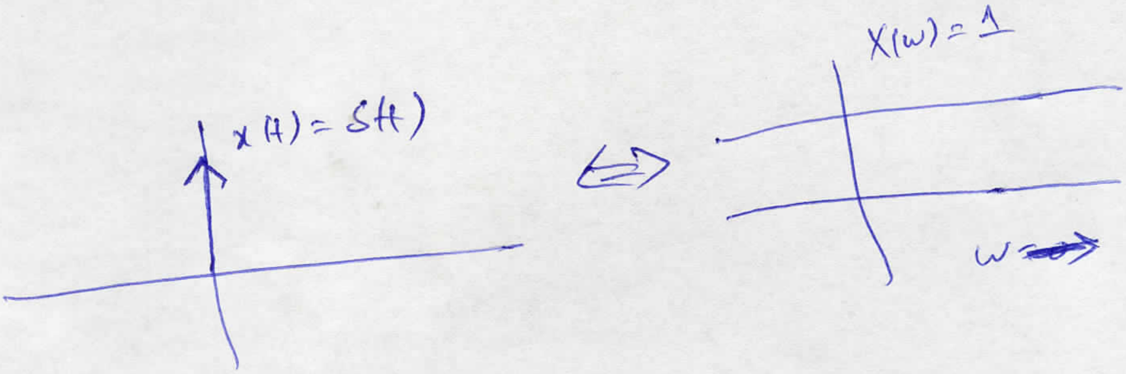
⑦

(f) Let's find FT of $\delta(t)$

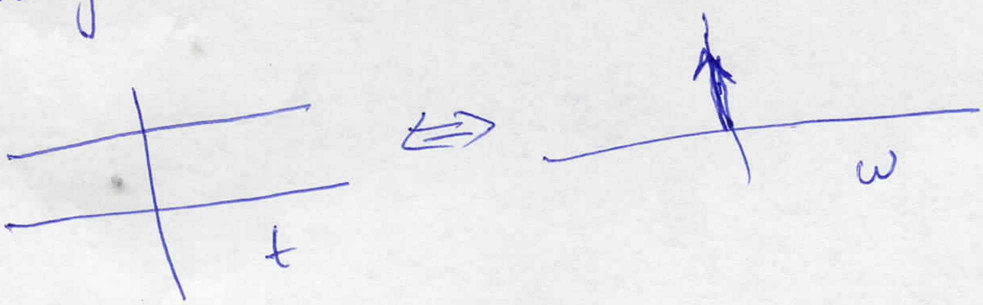
$$x(t) = \delta(t)$$

Sampling property of Impulse!

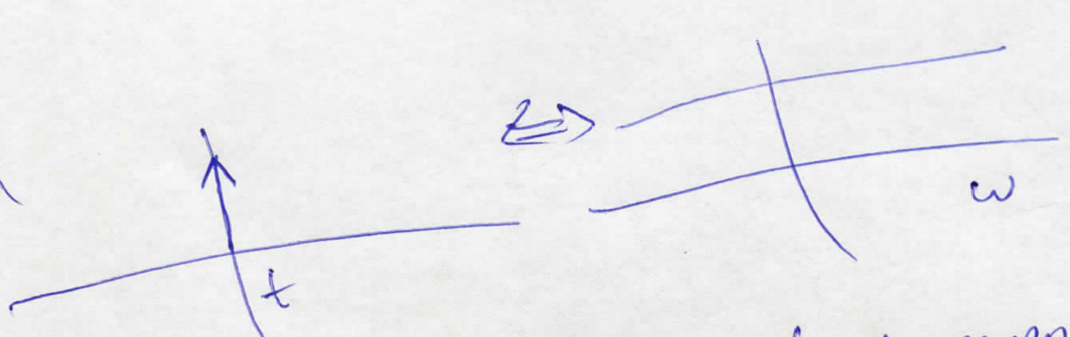
$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega(0)}\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



$\Rightarrow$ Duality note from (b) & (f) that



and



$\rightarrow$ In fact this two-way relation (with appropriate scaling by 2π) hols for all FT pairs! $\leftarrow$ called Duality

$$x(t) \Longleftrightarrow X(\omega)$$

Slides ← Lecture 21 & 22 → Practice FT

(1) Show that if $x(t) \Longleftrightarrow X(w)$

then $x(t-t_0) \Longleftrightarrow e^{jwt_0} X(w)$

Soln:
$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) e^{-jwt} dt$$

→ by change of variable $\tau = t - t_0$, we get

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(\tau) e^{-jw(\tau+t_0)} d\tau$$

$$= e^{-jwt_0} \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{-jw\tau} d\tau}_{} = e^{-jwt_0} X(w)$$

(2) Show that if $x(t) \Longleftrightarrow X(w)$

then $x(t) e^{jw_0 t} \Longleftrightarrow X(w-w_0)$

Soln
$$\mathcal{F}\{x(t) e^{jw_0 t}\} = \int_{-\infty}^{\infty} x(t) e^{jw_0 t} e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(w-w_0)t} dt = X(w-w_0)$$

3 Verify the duality property i.e

if $x(t) \Leftrightarrow X(\omega)$

then $X(t) \Leftrightarrow 2\pi x(-\omega)$

Soln By Defn. of IFT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

$$\Rightarrow \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = 2\pi x(t)$$

→ replacing t with $-t$, we get

$$\int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega = 2\pi x(-t)$$

→ Now interchanging t and ω, we get

$$\int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = 2\pi x(-\omega)$$

$$\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$$

$$\Rightarrow X(t) \Leftrightarrow 2\pi x(-\omega)$$

4 [using properties]

Find FT of $x(t) = \dfrac{1}{a^2+t^2}$

→ we notice from FT Table that

$$x(t) \quad e^{-a|t|} \Longleftrightarrow \frac{2a}{a^2+w^2} \quad X(s)$$

→ By Duality (i.e $X(t) \Longleftrightarrow 2\pi x(-w)$)

i.e → replace w with t on RHS, and replace t with $-w$ on LHS and multiply by

$$\Rightarrow X(t) = \frac{2a}{a^2+t^2} \qquad \left(\text{since } X(s) = \frac{2a}{a^2+w^2}\right)$$

and $\Rightarrow$ $x(-w) = e^{-a|w|}$ $\qquad \left(\text{since } x(t) = e^{-a|t|}\right)$

Duality gives: $\dfrac{2a}{a^2+t^2} \Longleftrightarrow 2\pi e^{-a|w|}$

By linearity : $\dfrac{1}{a^2+t^2} \Longleftrightarrow \dfrac{\cancel{2}\pi}{\cancel{2}a} e^{-a|w|}$

5 Verify that if $x(t) \Leftrightarrow X(s)$

then $\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$

Soln using IFT formula

$$\boxed{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{X(\omega)} e^{j\omega t} dW \quad — \quad ☆$$

$$\Rightarrow \frac{dx(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{\partial}{\partial t}(e^{j\omega t}) d\omega$$

$$\boxed{\frac{dx(t)}{dt}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{j\omega X(\omega)} e^{j\omega t} d\omega \quad — \quad \Delta$$

comparing ☆ & Δ tells us that

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$$

(6) Find IFT of $X(\omega) = \dfrac{1}{(a+j\omega)^2}$

using Time convolution theorem.

Soln Time convolution theorem says

if $y(t) = x(t) * h(t)$

then $Y(\omega) = X(\omega) H(\omega)$

Now we have $X(\omega) = \dfrac{1}{(a+j\omega)^2} = \underbrace{\dfrac{1}{(a+j\omega)}}_{X_1(\omega)} \underbrace{\dfrac{1}{(a+j\omega)}}_{X_2(\omega)}$

$$\Rightarrow \quad X(\omega) = X_1(\omega) X_2(\omega)$$

$$\Rightarrow \quad x(t) = x_1(t) * x_2(t)$$

where $x_1(t) = x_2(t) = e^{-at} u(t)$ (since $e^{-at}u(t) \Leftrightarrow \dfrac{1}{a+j\omega}$)

$$\Rightarrow \quad x(t) = e^{-at}u(t) * e^{-at}u(t)$$

(recall $f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$)

$$= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) e^{-a(t-\tau)} u(t-\tau) d\tau$$

$$= e^{-at} \int_0^t d\tau = t e^{-at} u(t)$$

7. Assuming zero ICs, find impulse response of

$$\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

Soln

Taking FT

$$\Rightarrow \quad j\omega Y(\omega) + 2Y(\omega) = X(\omega) + j\omega X(\omega)$$

$$\Rightarrow \quad (j\omega + 2)\, Y(\omega) = (1 + j\omega)\, X(\omega)$$

$$Y(\omega) = \underbrace{\frac{(1+j\omega)}{(2+j\omega)}}_{H(\omega)} X(\omega)$$

$$\Rightarrow H(\omega) = \frac{1+j\omega}{2+j\omega} = \frac{2+j\omega-1}{2+j\omega} = 1 - \frac{1}{2+j\omega}$$

$\Rightarrow$ Taking IFT to get

$$h(t) = \delta(t) - e^{-2t}u(t).$$

8 Proove Parseval's Theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

→ Reading assignment: look for its proof & usage!

9 Given System with zero-state

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

find $y(t)$ if $x(t) = e^{-t} u(t)$

Soln.

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2+j\omega}$$

also $X(\omega) = \dfrac{1}{1+j\omega}$

$$\Rightarrow Y(\omega) = H(\omega) X(\omega) = \frac{1}{1+j\omega} \frac{1}{2+j\omega}$$

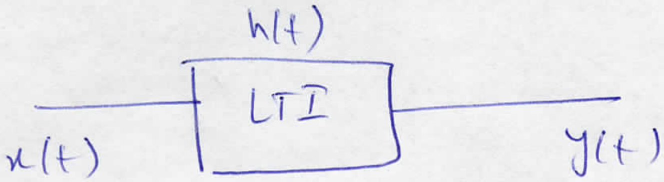
with PFE: $y(t) = (e^{-t} - e^{-2t}) u(t)$

Lecture 23

Wrap up introduction to Fourier with the final three points.

1. What happens to frequencies as they pass through LTI systems?
2. Fourier series: spectra of periodic signals.
3. Link b/w Z, $\mathcal{L}$, and $\mathcal{F}$.

1



$\Rightarrow$ wkt: $y(t) = x(t) * h(t)$

Take FT $\quad Y(\omega) = X(\omega)\,H(\omega)$ $\quad \left(\begin{array}{l}\text{recall} \\ H(\omega) = \text{complex} \\ \quad = |H(\omega)|\, e^{j\angle H(\omega)}\end{array}\right)$

$\rightarrow$ so $H(\omega)$ acts as a scaling and phase shifting in the frequencies of $x(t)$ (~~characterized~~ represented by $X(\omega)$)

$\rightarrow$ To see this, write $H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$

and $X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)}$

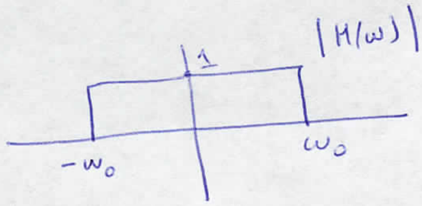
$\rightarrow$ Then output is $Y(\omega) = \underbrace{|H(\omega)|\,|X(\omega)|}_{\text{amplitude scale}}\; e^{j\overbrace{\angle X(\omega) + H(\omega)}^{\text{phase shifter}}}$

$|Y(\omega)| \qquad \angle Y(\omega)$

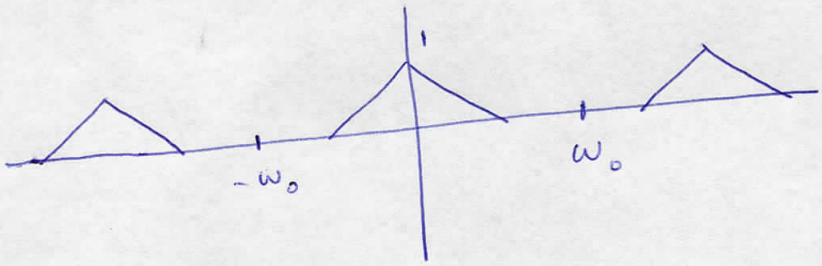
$\rightarrow H(\omega) =$ Transfer function, also called $\boxed{\text{Frequency Response}}$ of the system as it tells what the system will do to frequencies.

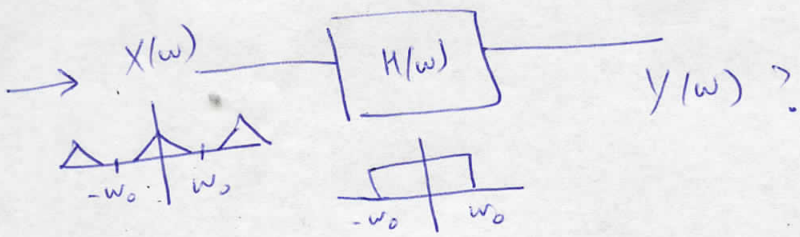
e.g. → Let's say there is a system that behaves as
ex 1
a filter, saying I will only allow frequencies below ω_0 to pass and will block all others ("Low Pass Filter")

→ The $|H(\omega)|$ of such a system could look like



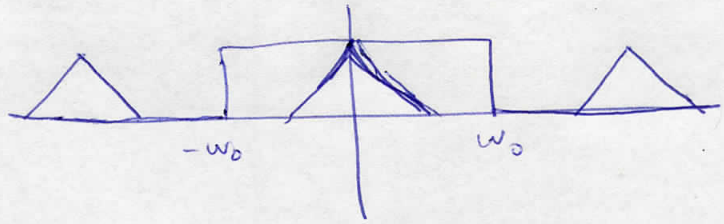
→ Now what happens to a signal that has frequencies both above and below ω_0?, e.g. let's say the signal frequencies are





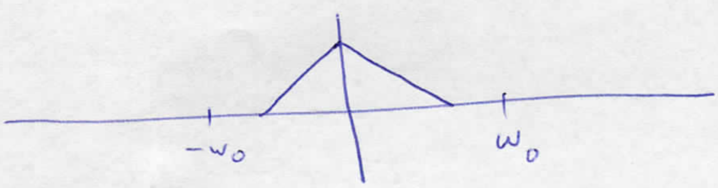
→ since $|Y(\omega)| = |X(\omega)||H(\omega)|$, we get

$|X(\omega)||H(\omega)| \Rightarrow$



$= Y(\omega)$

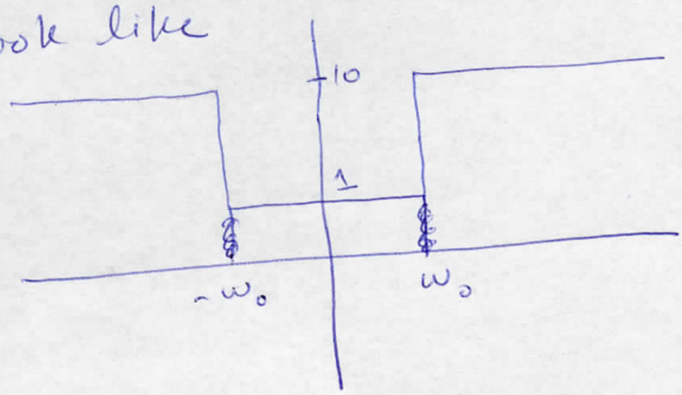
$\Rightarrow$



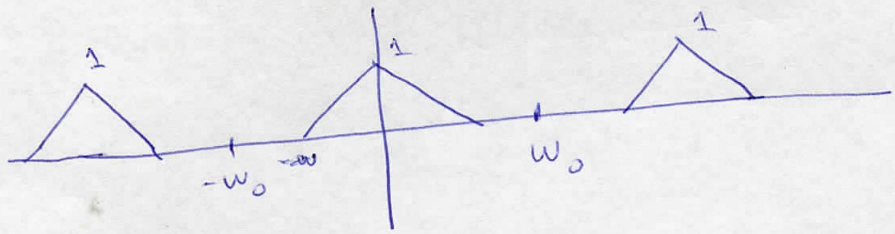
→ i.e. only low frequency part remains after filtering.

ex 2 → How about a system that says I will let frequencies below ω_o pass through unchanged but will amplify frequencies above ω_o by 10x ?

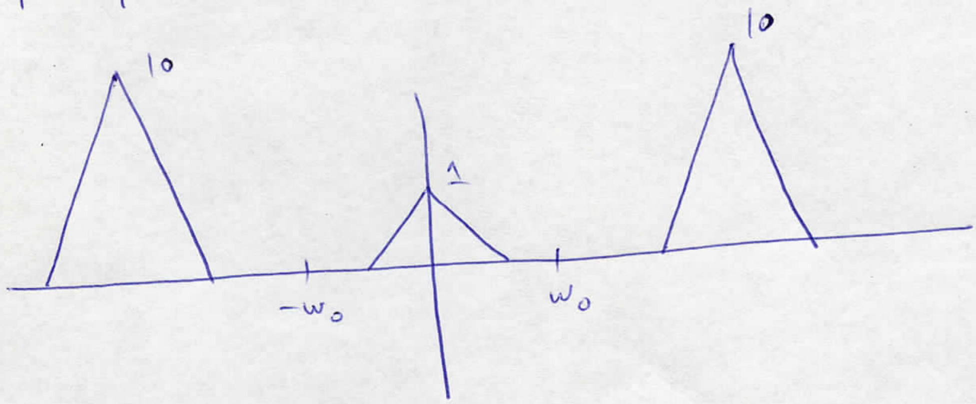
→ The magnitude freq. resp. $|H(\omega)|$ of such system (selective amplifier ?) may look like



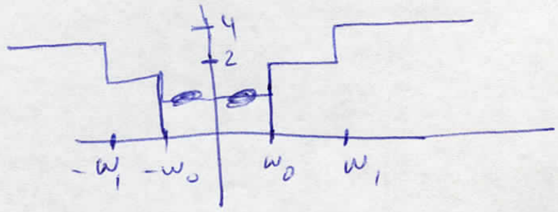
→ now what happens to freqs of $X(\omega)$ with $|X(\omega)|$



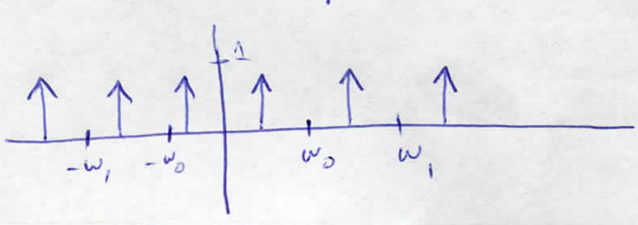
we get → $|Y(\omega)| = |X(\omega)|\,|H(\omega)|$



ex 3 Given $|H(\omega)|$

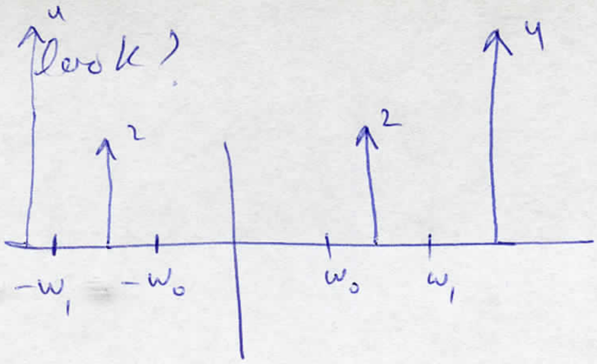


and $|X(\omega)|$

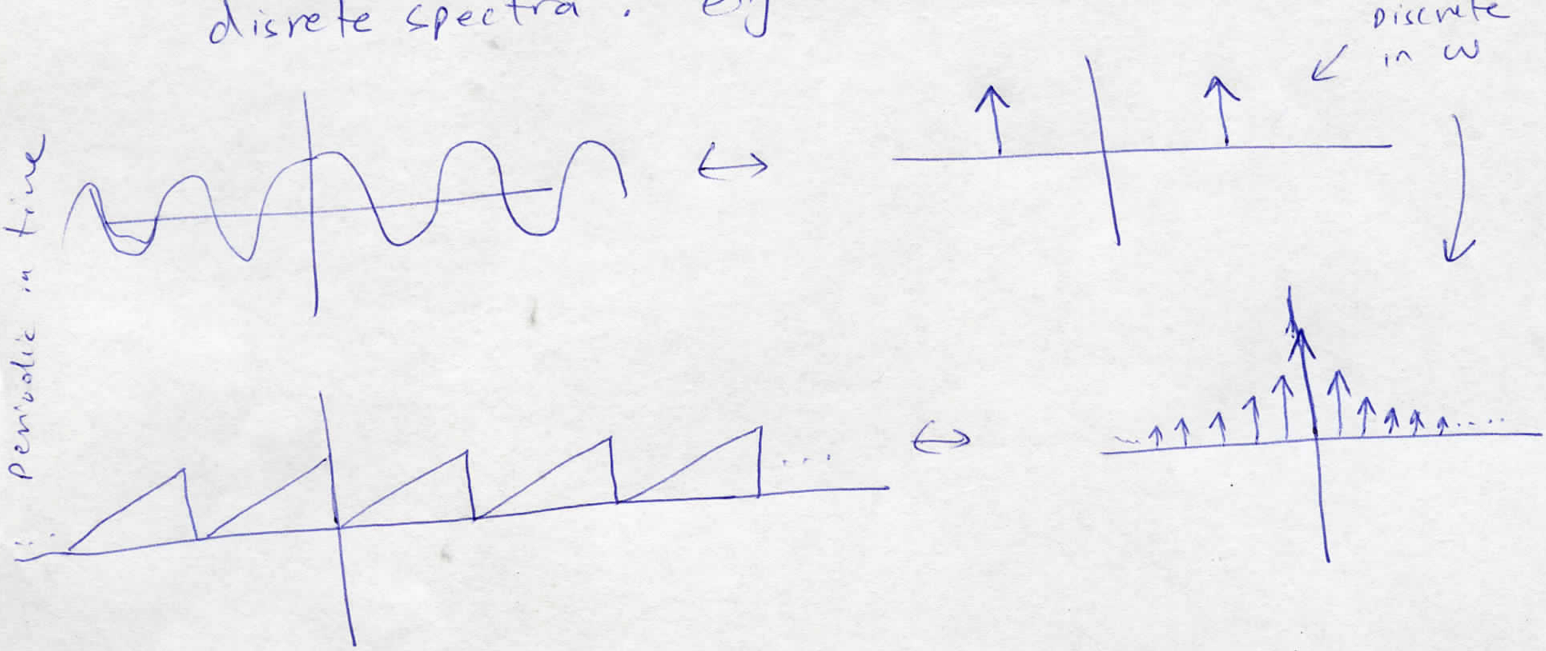


How should $|Y(\omega)|$ look?

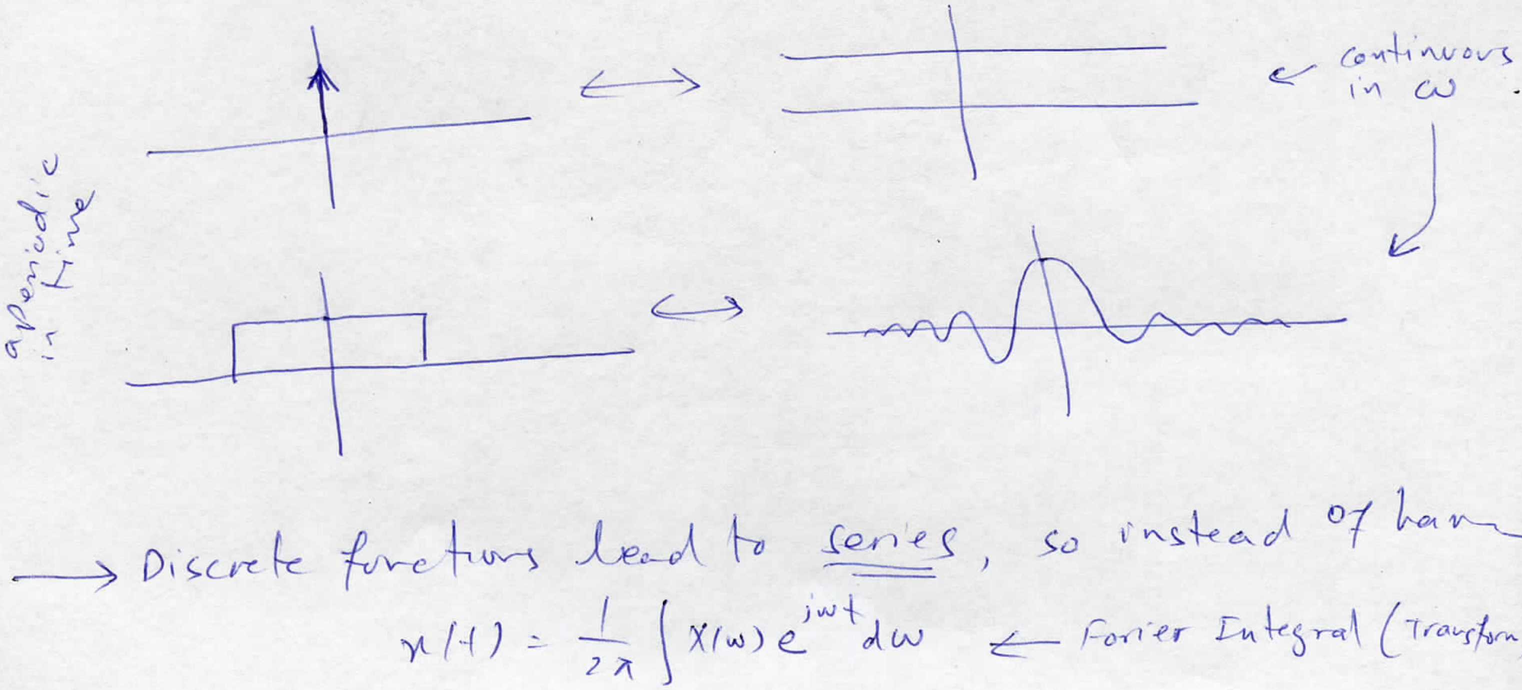
$\rightarrow$ $|Y(\omega)|$ $\Rightarrow$



(2) Previously we saw that: "Periodic signals have disrete spectra". e.g.



whereas aperiodic signals have continuous spectra, e.g.



$\longrightarrow$ Discrete functions lead to series, so instead of having

$$x(t) = \frac{1}{2\pi}\int X(\omega)\, e^{j\omega t}\, d\omega$$ $\leftarrow$ Forier Integral (Transform)

we can have something like

$$x(t) = \sum_{\forall n} X_n e^{jn\omega t} \quad \leftarrow \text{Fourier } \underline{\text{Series}}$$

→ In fact, it can be shown that in all practical cases a periodic signal with fundamental period T_0 and angular frequency $\omega_0 = \frac{2\pi}{T_0}$ can be written as a sum of sinusoids of frequencies ω_0 and its multiples $(2\omega_0, 3\omega_0, 4\omega_0 \ldots)$, i.e it can be written in terms of

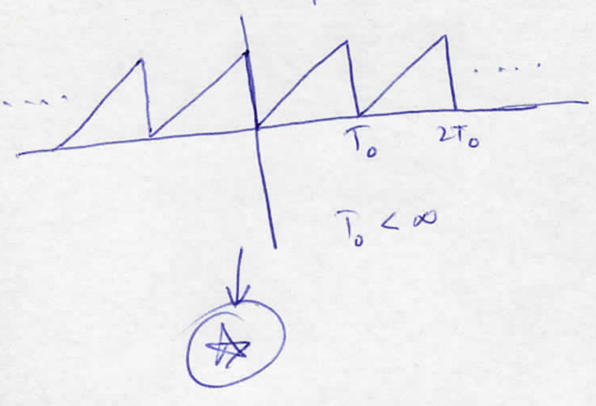
$$e^{j\omega_0 t},\ e^{j2\omega_0 t},\ e^{j3\omega_0 t},\ \ldots\ e^{jn\omega_0 t},\ \ldots$$

→ This is called the Exponential Fourier series, and it says that $x(t)$ is periodic with fundamental frequency ω_0 then we can write it as

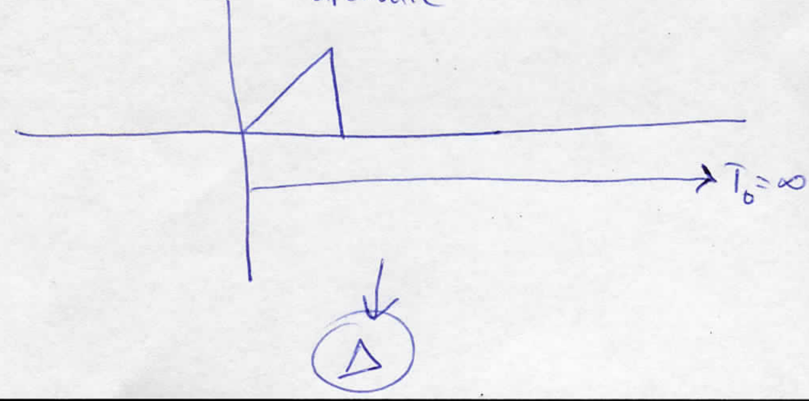
$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{———} \; ☆$$

→ In fact it can be shown that as $T_0 \to \infty$ (i.e signal becomes aperiodic) ☆ becomes the Fourier Transform (integral) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$ — Δ

periodic



aperiodic



→ Furthermore, we can also show that $x(t)$ with fund. freq. w_0 can be written in terms of $\cos(nw_0t)$ and $\sin(nw_0t)$ as

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nw_0t) + b_n \sin(nw_0t)\right)$$

→ This representation is called
"Trigonometric Fourier Series".

→ we will not touch upon those in further detail.

3 $\mathcal{Z}, \mathcal{L}, \mathcal{F}$

we have seen that $\mathcal{Z}$, $\mathcal{L}$, and $\mathcal{F}$ use bases function

Z^n, e^{st}, and e^{jwt} respectively with (z & s complex numbers)

such that

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \qquad s = \sigma + jw$$

$$X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

→ How are they related?

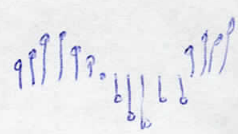
$z = e^{sT}$

Laplace

Discrete signal case with signal sampled at period T

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] e^{-snT}$$

ingredients:



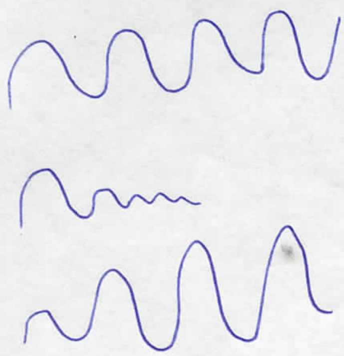
etc.

Ingredients limited to pure complex sinusoids only $e^{-j\omega t}$ $\left(\substack{s=j\omega \\ \sigma = 0}\right)$ (*) — Fourier Tx.

- continuous signals
- uses complex exponentials allowing for pure, growing, and decaying sinusoids as

$$e^{-st} = e^{-\sigma t} e^{-j\omega t}$$

Ingredients:



Ingredients:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

(*) Note that even though we may get FT from LT by setting $s = j\omega$, the two Tx's may have different ROCs.

ES 332
Signals and Systems

with

Dr. Naveed R. Butt

@

GIKI - FES

In the last few months, we have seen…

What is a signal and what is a system? (from engineering mathematics perspective)

Chapter 1

What are some of the common types of signals and systems?

Chapter 1

What are some of the useful properties of signals and systems?

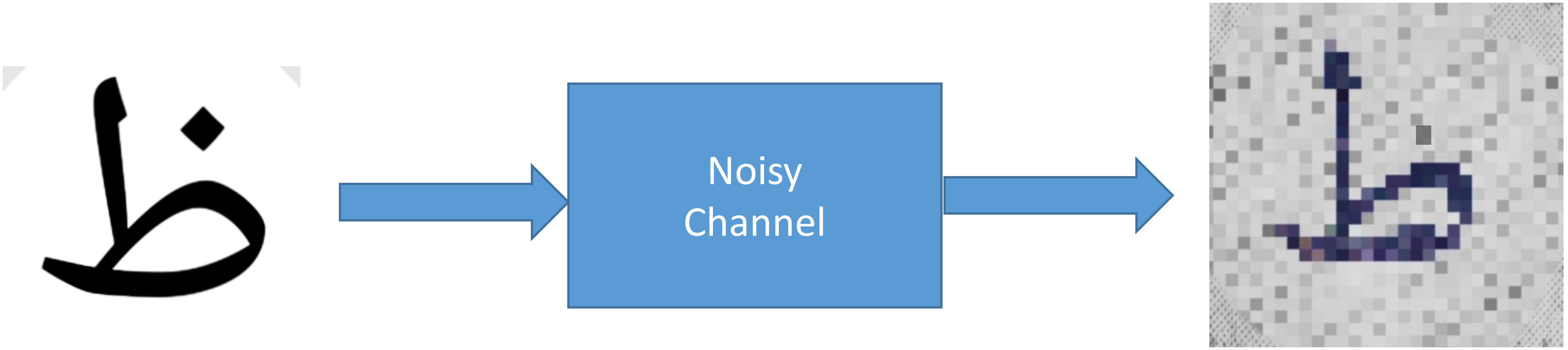
Chapter 1

What are some of the major ways of modelling and analyzing signals and systems?

Time Domain	**Chapters 2, 3**
Laplace	**Chapter 4**
Z-Transform	**Chapter 5**
Fourier	**Chapters 6, 7**

How do signals and systems interact?

Chapters 2-7


Noisy
Channel

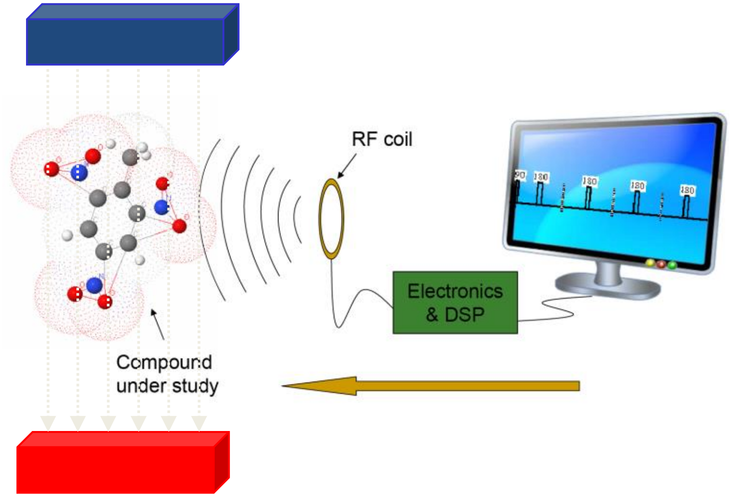
Let us now look at some practical applications of what we've learned…

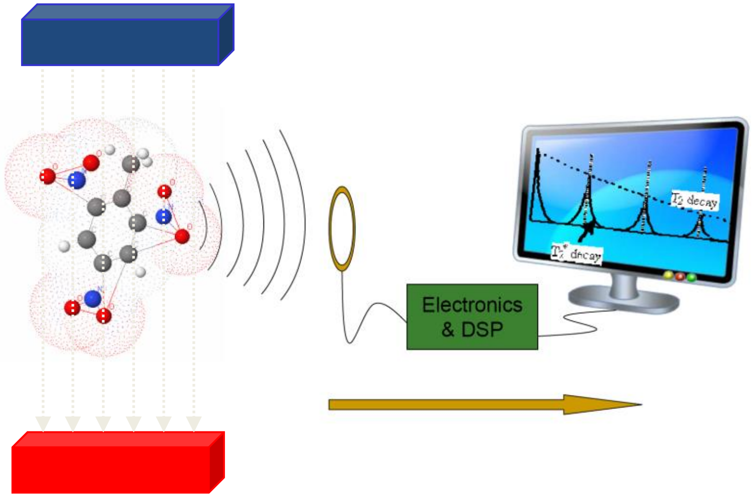
- **Examples from my own research**: Material Identification
- **Analyzing circuits** : Laplace Transform
- **Wireless Communications** : Z Transform
- **Filter Design** : Fourier Transform

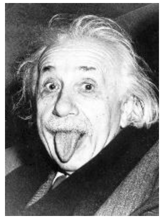
CONPHIRMER Project

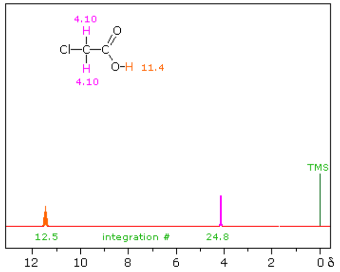
Question: how to quickly tell whether a medicine is fake?

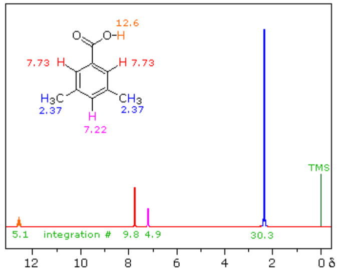
NMR vs. NQR


RF coil
Electronics & DSP
Compound under study

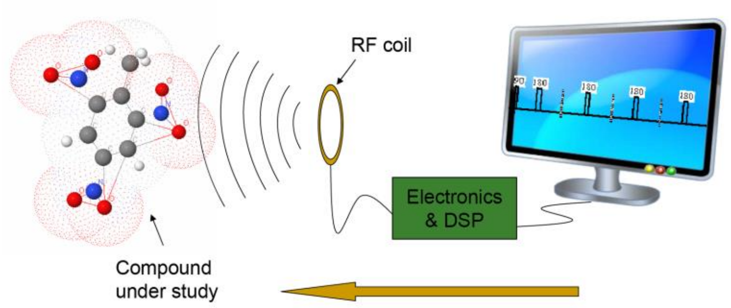

Electronics & DSP




TMS
12.5
integration #
24.8


TMS
integration #
30.3

NQR Signal


RF coil
Electronics
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Compound
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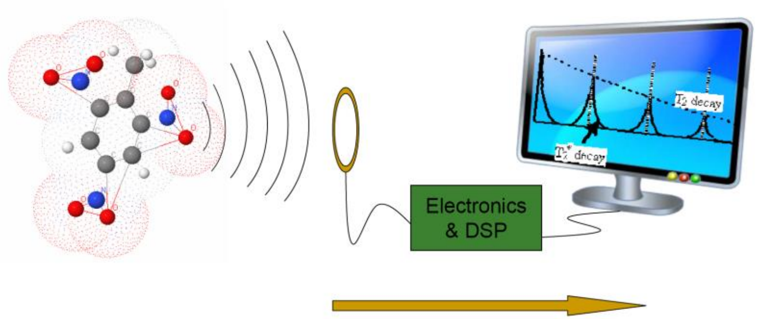
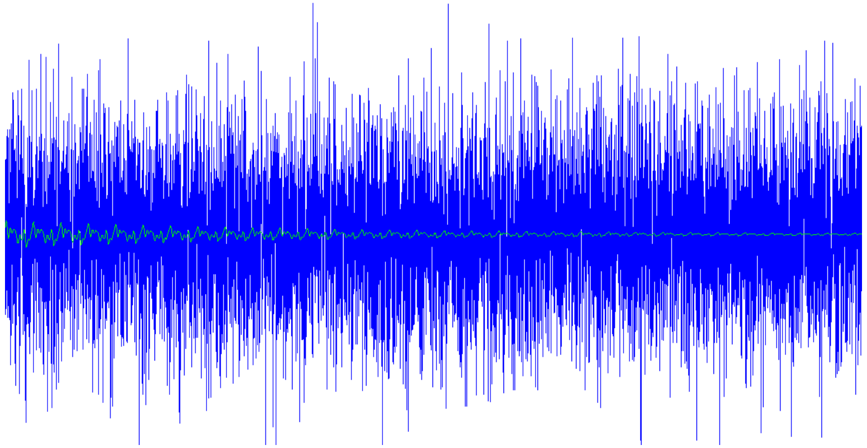
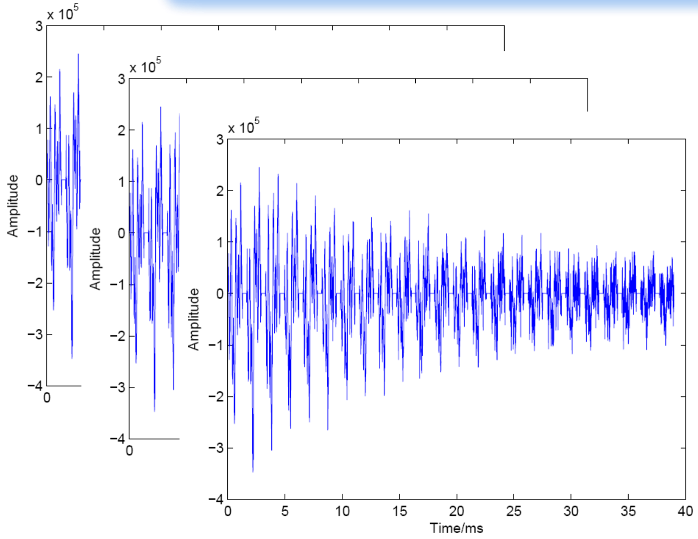
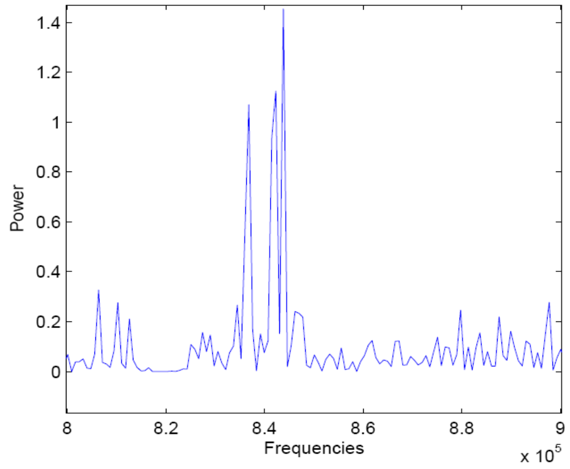

Electronics
& DSP



NQR Signal Modeling

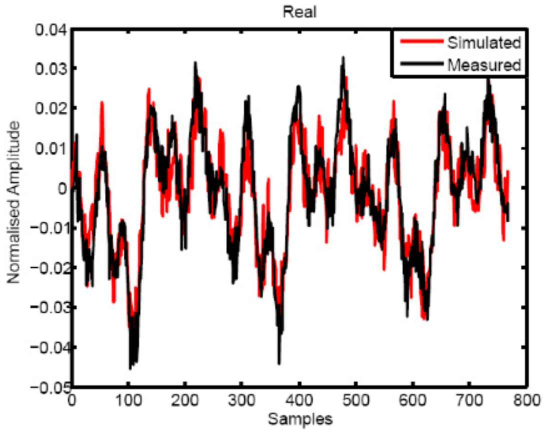




$$y_m^{(p)}(t) = \sum_{k=1}^{d^{(p)}} \alpha_k^{(p)} e^{-(t+m\mu)\eta_k^{(p)}} e^{-\beta_k^{(p)}|t-t_{sp}|+i\omega_k^{(p)}(T)t}$$

$$\omega_k^{(p)}(T) = a_k^{(p)} - b_k^{(p)} T$$

$$\alpha_k^{(p)} = \rho \kappa_k^{(p)} \qquad \left\| \boldsymbol{\kappa} - \boldsymbol{\kappa_a} \right\|_2^2 \leq \epsilon$$

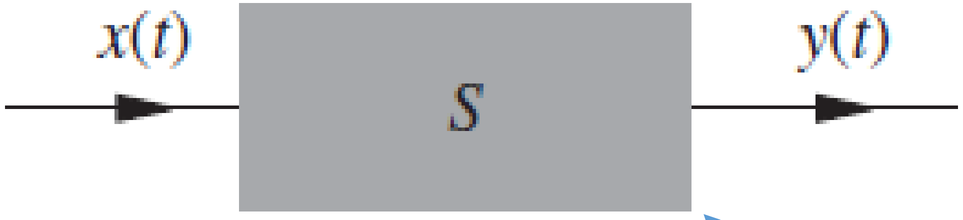


Let us now look at some practical applications of what we've learned…

- **Examples from my own research**: Material Identification
- **Analyzing circuits** : Laplace Transform
- **Wireless Communications** : Z Transform
- **Filter Design** : Fourier Transform

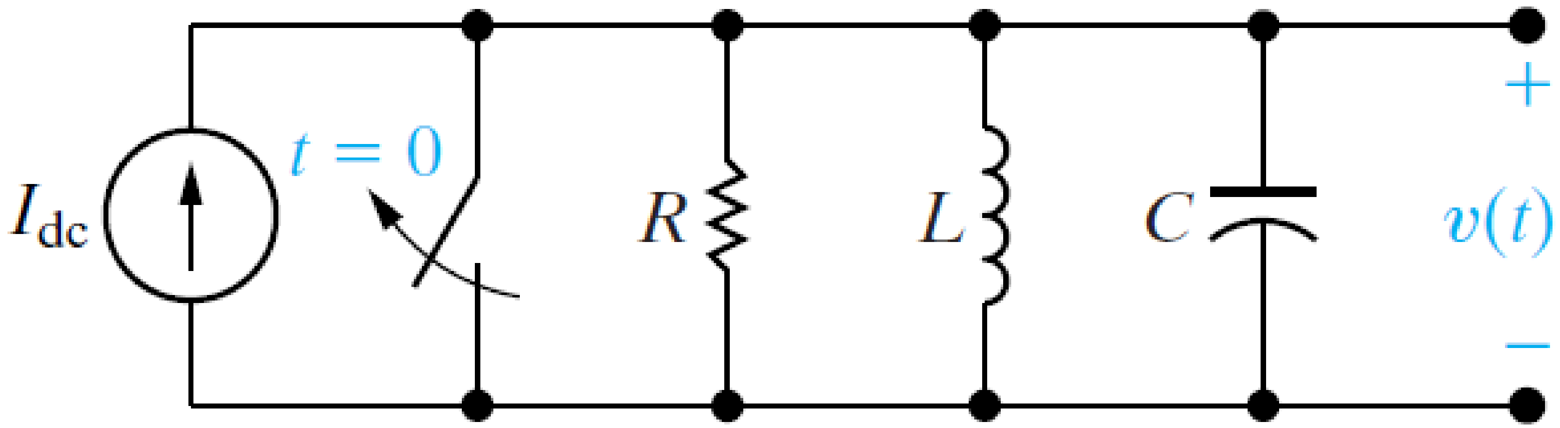
“System”

AN ENTITY THAT PROCESSES A SIGNAL

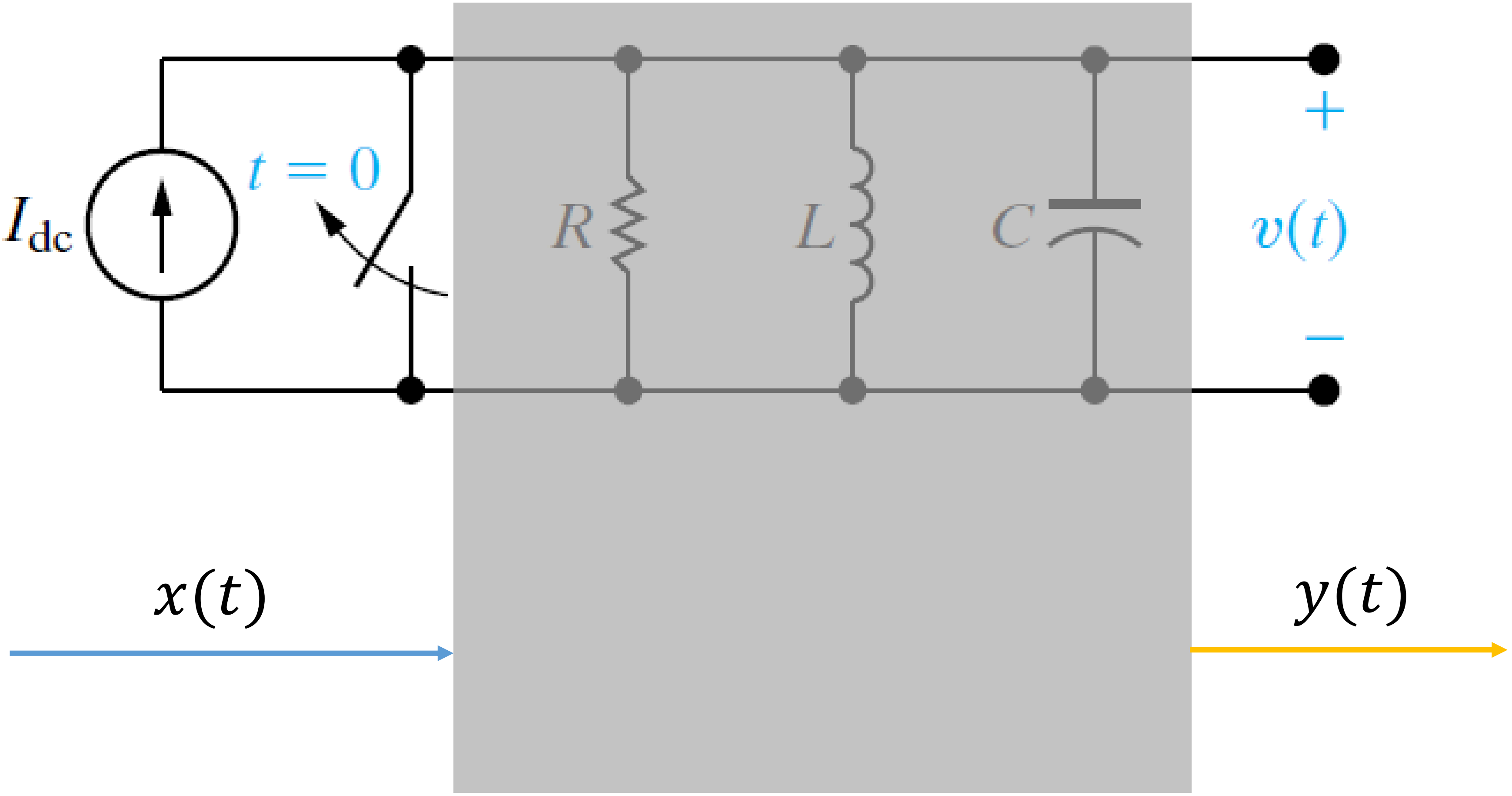


Can be something physical, or just an algorithm.

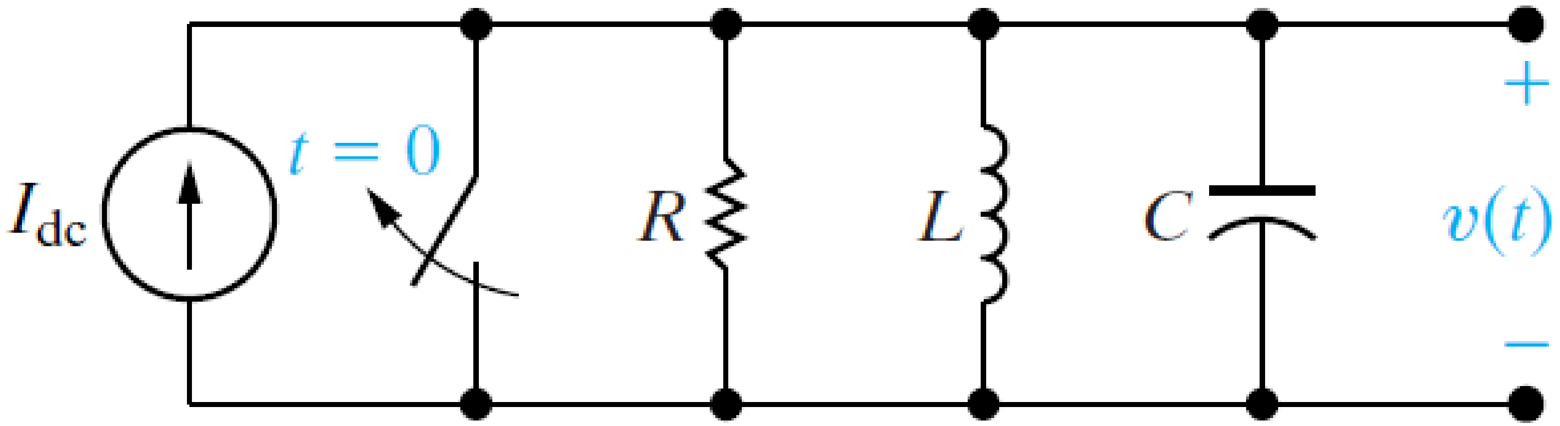
Example of a System



Example of a System



System Modeled as a Differential Equation (Time-Domain)



$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t).$$

Using Laplace (Frequency Domain) to Solve and Analyze the System

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{\mathrm{dc}}u(t).$$

$$\boxed{\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{\mathrm{dc}}\left(\frac{1}{s}\right),}$$

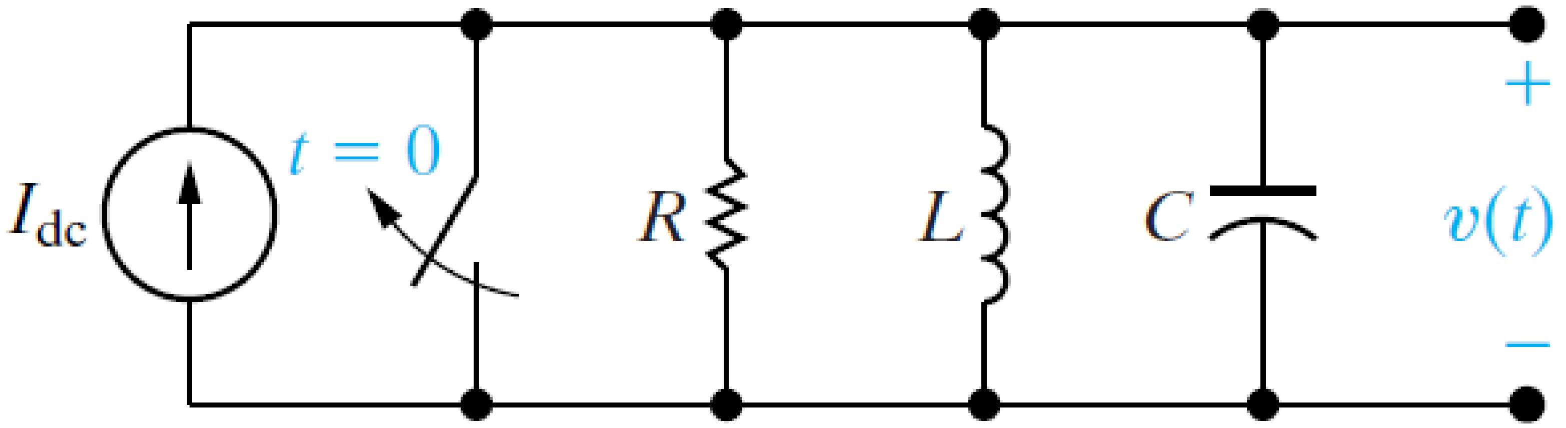
Using Laplace (Frequency Domain) to Solve and Analyze the System

$$\boxed{\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{\mathrm{dc}}\left(\frac{1}{s}\right),}$$

$$V(s) = \frac{I_{\mathrm{dc}}/C}{s^2 + (1/RC)s + (1/LC)}$$

Put values and use PFE to get inverse Laplace.

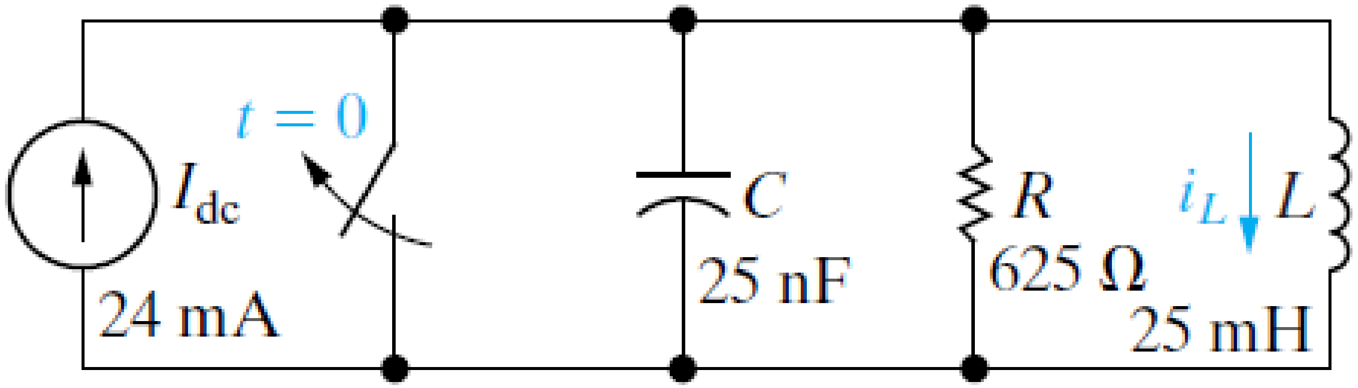
Using Laplace (Frequency Domain) to Solve and Analyze the System



$$v(t) = \mathscr{L}^{-1}\{V(s)\}$$

Analysis: what are the initial and final (steady-state) values of $i_L(t)$?

<u>Assuming zero ICs</u>



$$I_L = \frac{I_{dc}/LC}{s[s^2 + (1/RC)s + (1/LC)]}$$

$$i_L(\infty) = \lim_{s \to 0} sI_L(s) = I_{dc}$$

Do these values make sense?

$$i_L(0) = \lim_{s \to \infty} sI_L(s) = 0$$

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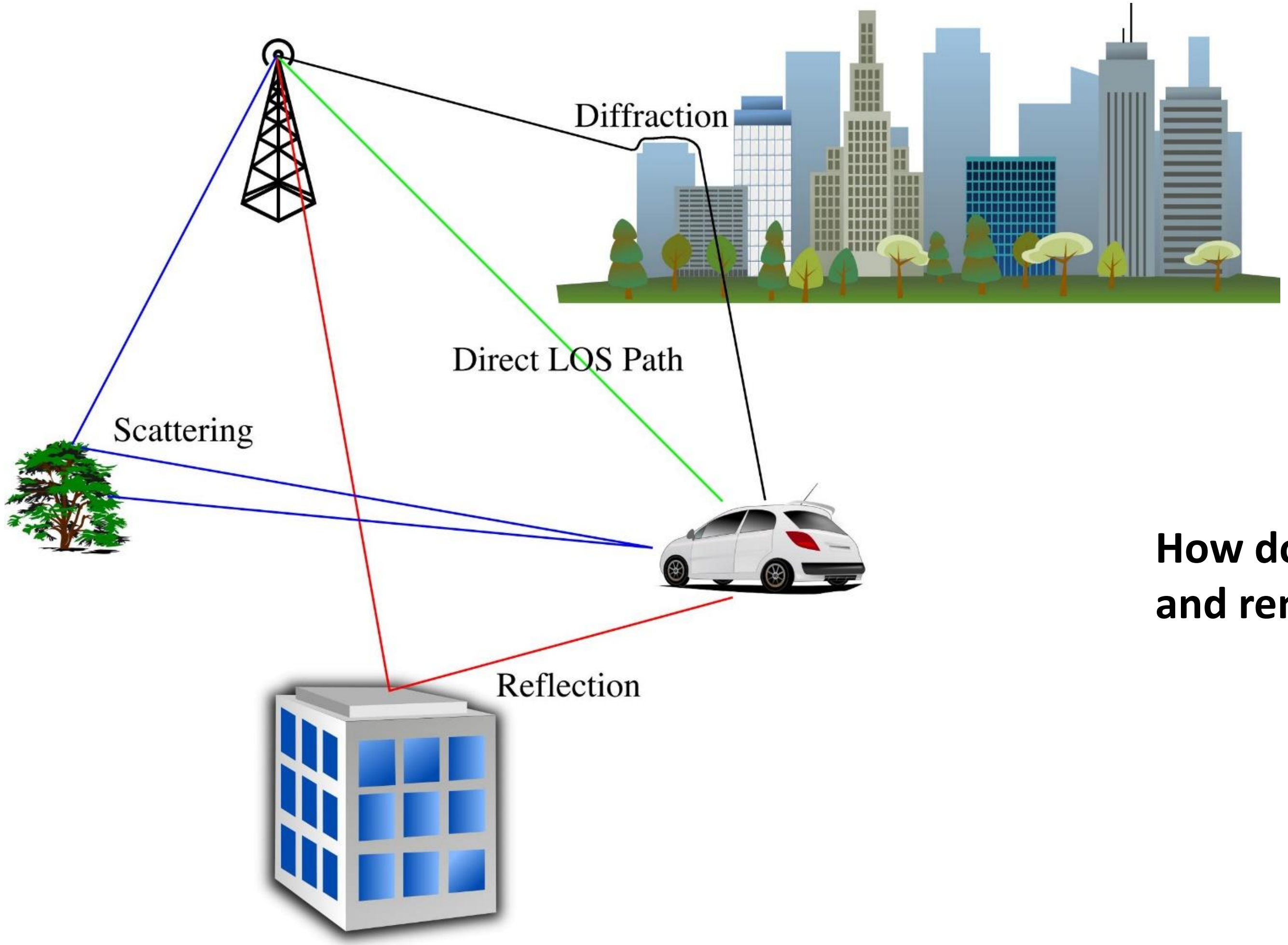
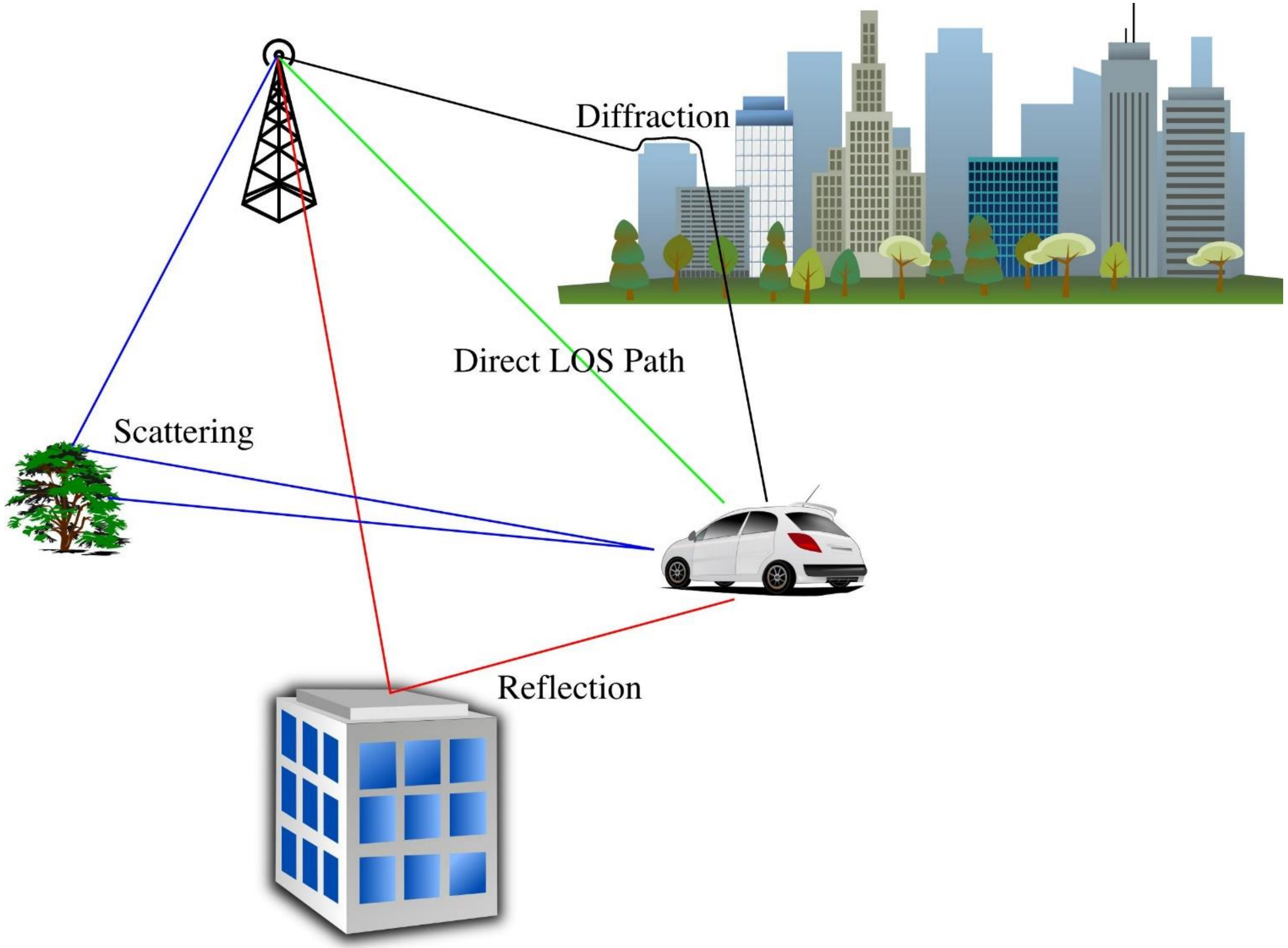
Multipath Signals in Wireless Communications



How do we model the signal (plus echoes) and remove the echoes?

Discrete-Time System Model

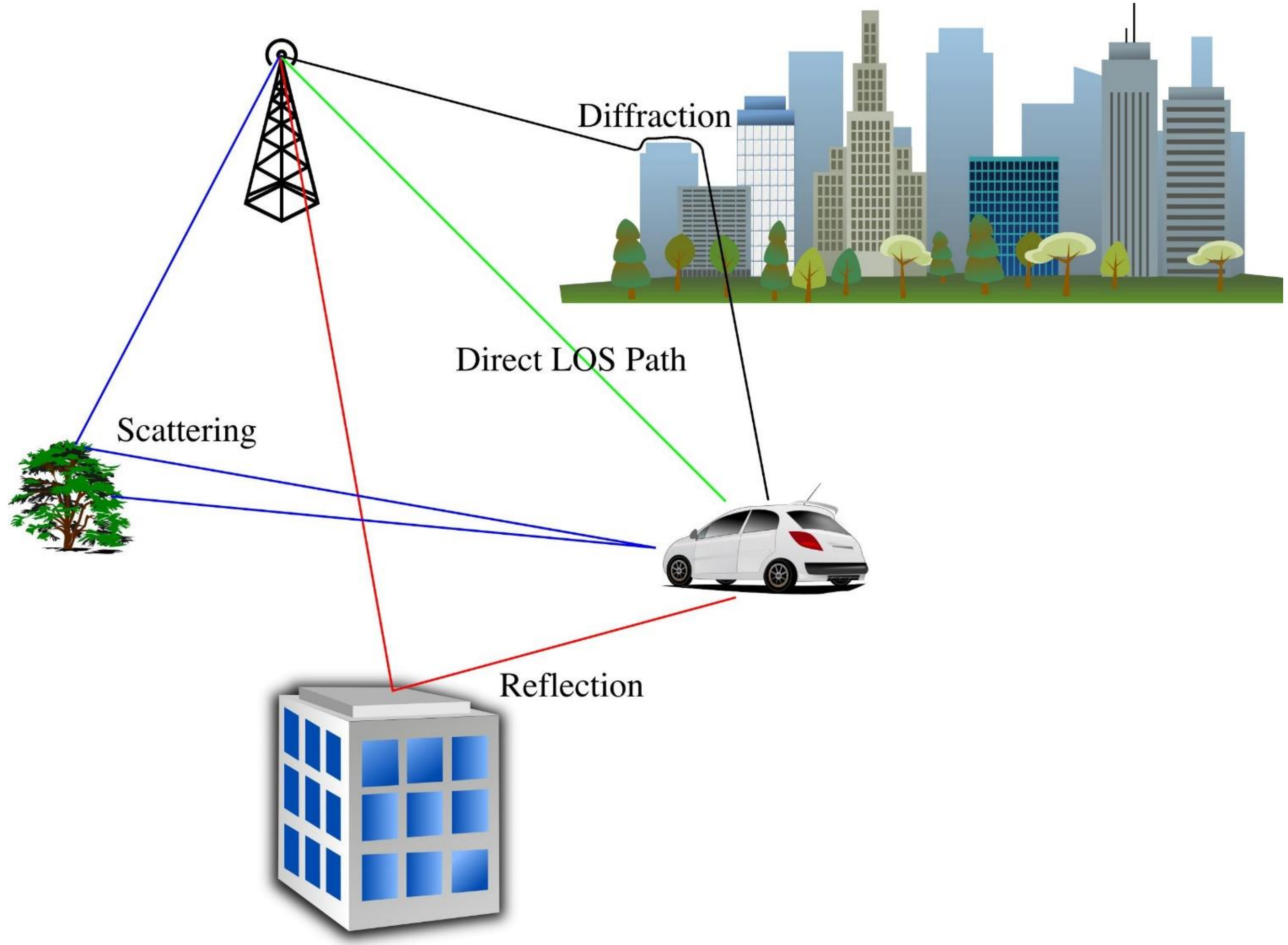


Perhaps only the green and red components are strong (and the rest are very weak)? If so, our model may look like:

$$y[n] = x[n] - \alpha x[n - N_0]$$

$$\alpha = 0.8, \; N_0 = 11$$

Using Z Transform (Frequency Domain) to Analyze and "Fix" the System



$$\alpha = 0.8, \; N_0 = 11$$

$$y[n] = x[n] - \alpha x[n - N_0]$$

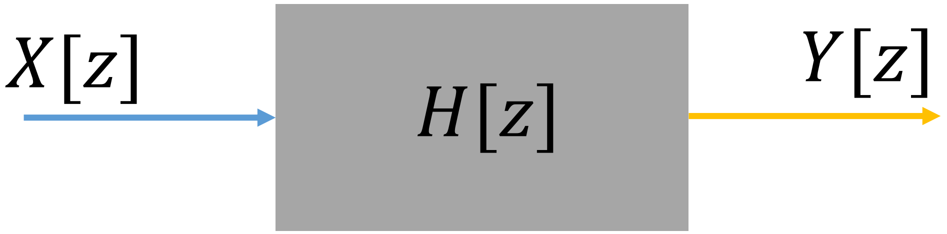
$$Y[z] = X[z] - 0.8Z^{-11}X[z]$$

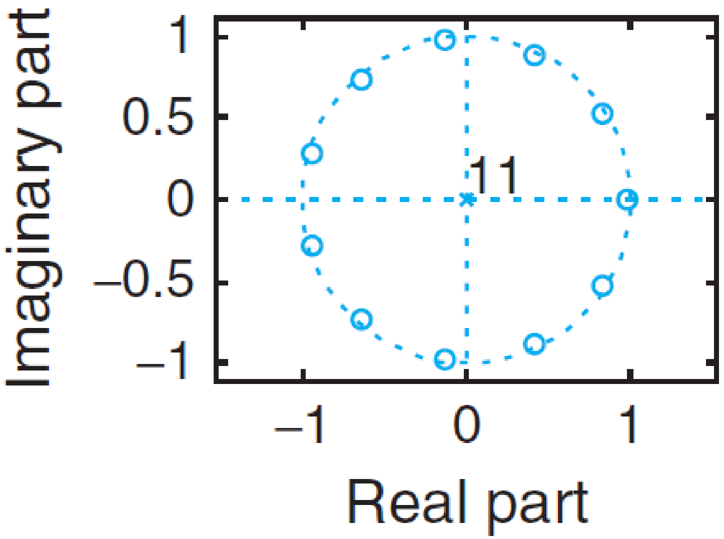
$$Y[z] = \underbrace{(1 - 0.8Z^{-11})}_{\text{"System" Transfer Function } H[z]} X[z]$$

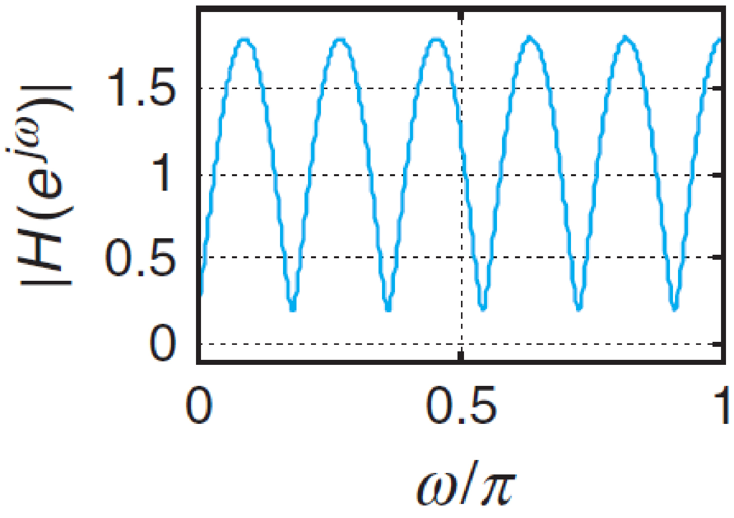
"System" Transfer Function $H[z]$

Using Z Transform (Frequency Domain) to Analyze and "Fix" the System

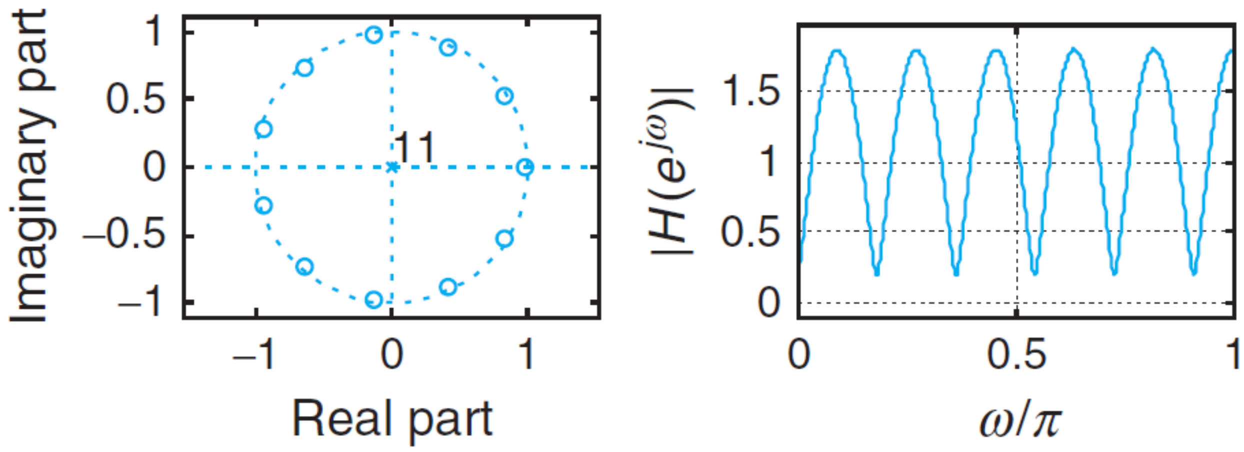
$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.8z^{-11} = \frac{z^{11} - 0.8}{z^{11}}$$







The environment behaves as a "comb" filter. **How may we remove this effect to get original signal back?**



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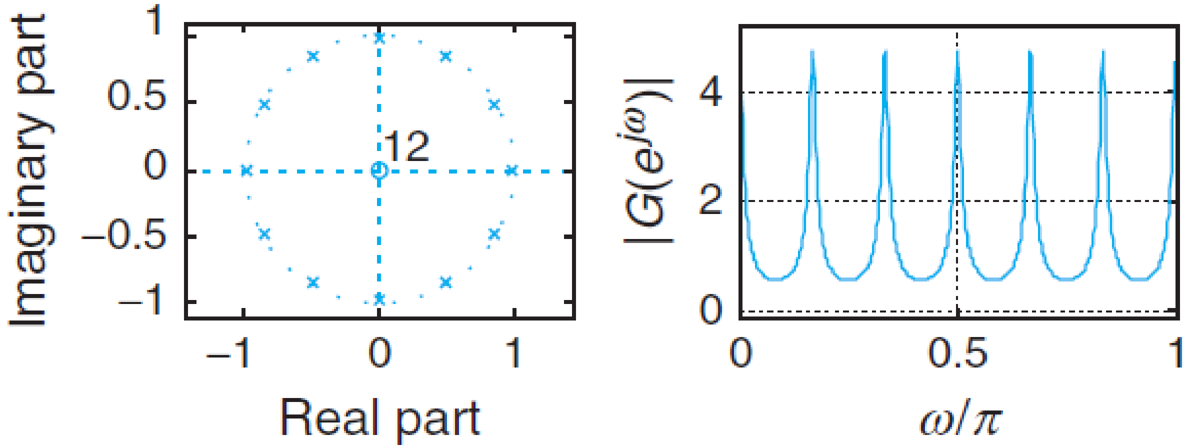
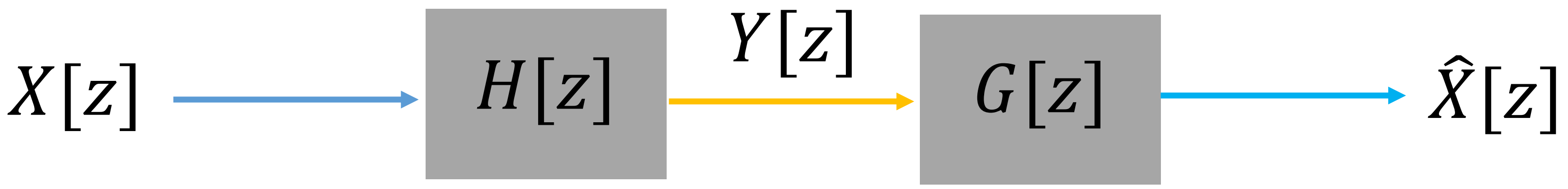


One way is to pass the received signal through a filter that cancels the zeros and poles of the system.

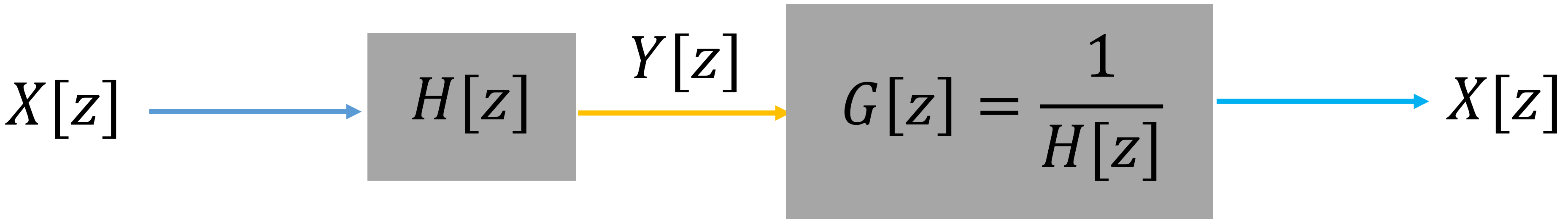


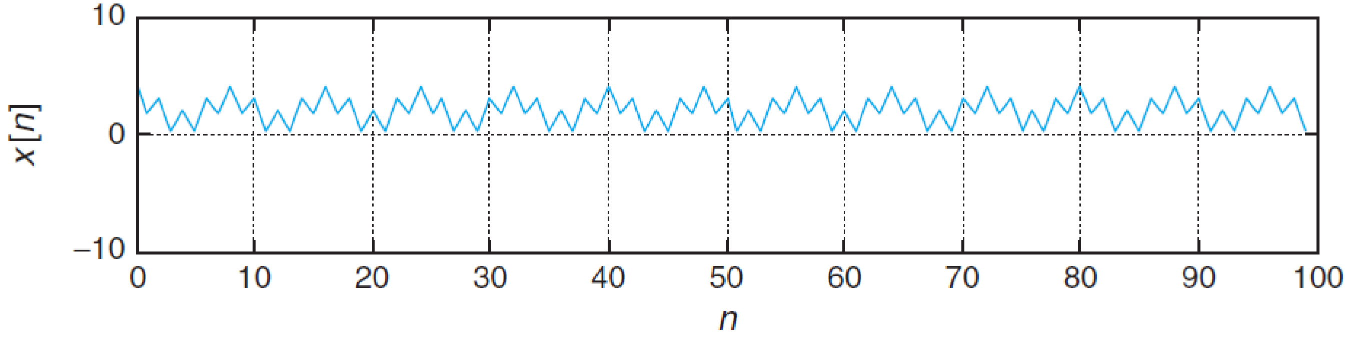
$$X[z] \longrightarrow \boxed{H[z]} \xrightarrow{Y[z]} \boxed{G[z]} \longrightarrow \hat{X}[z]$$

To get $\hat{\boldsymbol{X}}[\boldsymbol{z}] = \boldsymbol{X}[\boldsymbol{z}]$ we must have $\boldsymbol{H}[\boldsymbol{z}]\boldsymbol{G}[\boldsymbol{z}] = \mathbf{1}$, which gives:

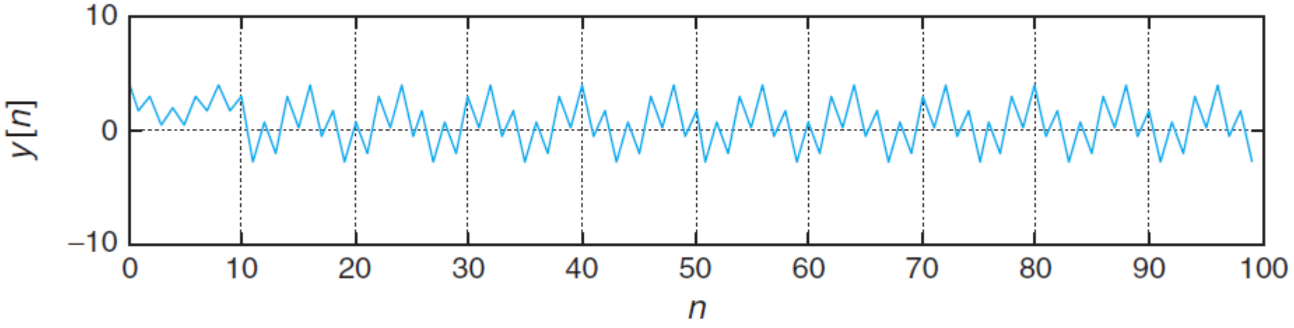
$$G(z) = \frac{z^{11}}{z^{11} - 0.8}$$

This filter will remove the effect of multipath and give our original signal back.





Example signals.



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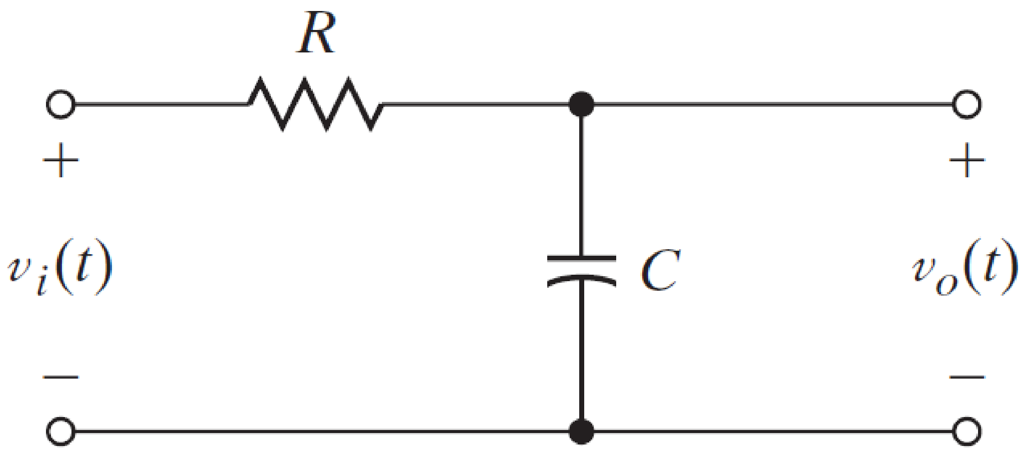
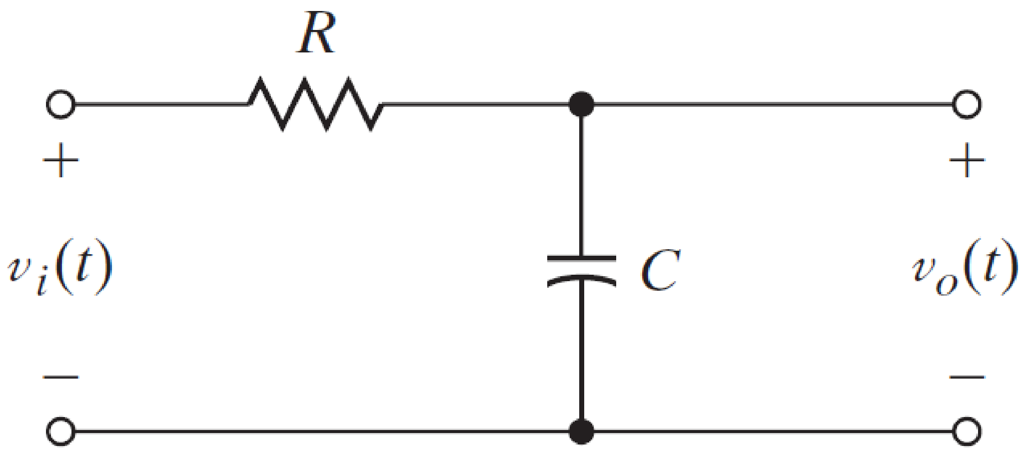
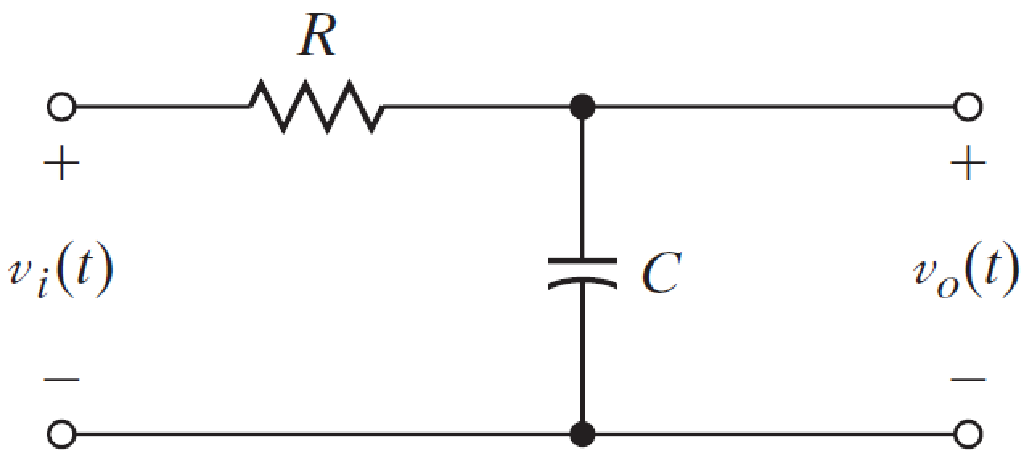
Example of a Low Pass Filter



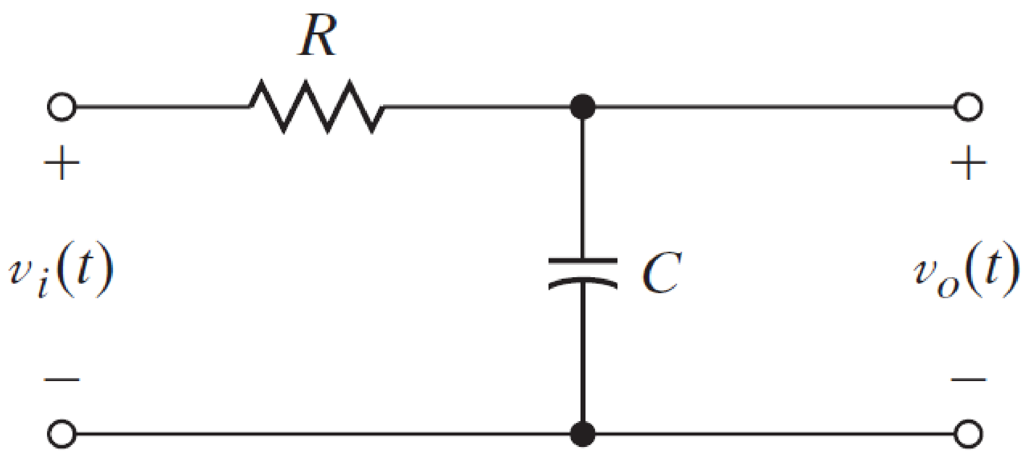
Using Fourier Transform Determine How the Filter's Cutoff Frequency is Linked to R and C





$$v_i(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau; \quad v_o(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$V_i(\omega) = RI(\omega) + \frac{1}{j\omega C} I(\omega), V_o(\omega) = \frac{1}{j\omega C} I(\omega)$$



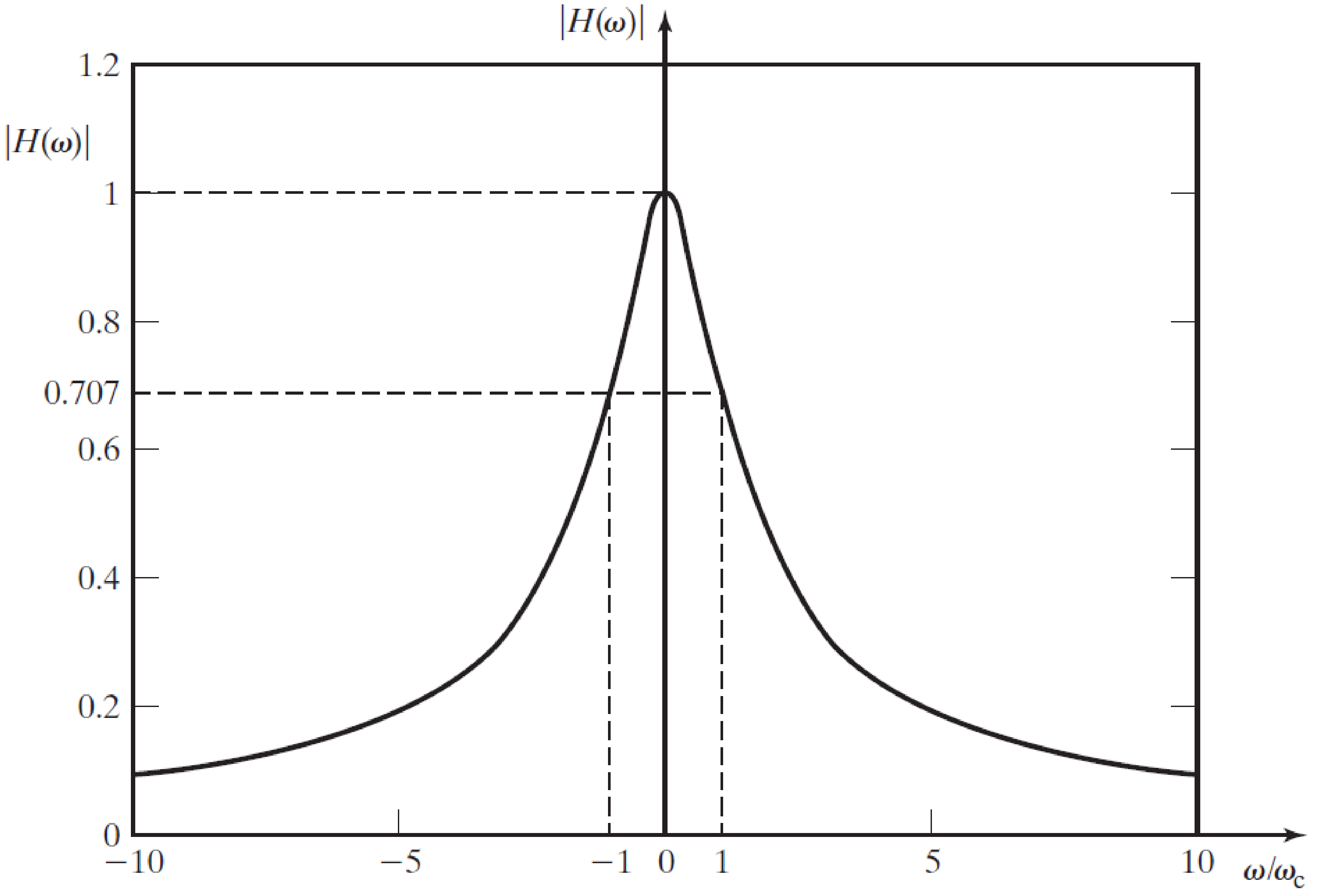
$$v_i(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau; \quad v_o(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$V_i(\omega) = RI(\omega) + \frac{1}{j\omega C}I(\omega), V_o(\omega) = \frac{1}{j\omega C}I(\omega)$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1}{1 + j\omega RC}.$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}$$

$$\omega_c = \frac{1}{RC}$$



How should you change R and C if you would like to increase (or decrease) the cut-off frequency?

Questions?? Thoughts??